IMPACT ON RAILWAY BRIDGES

BY

CHARLES THOMAS GEORGE LOONEY
B.S., Carnegie Institute of Technology, 1932
M.S., University of Illinois, 1934

THESIS
SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
ENGINEERING IN THE GRADUATE SCHOOL
OF THE UNIVERSITY OF ILLINOIS, 1940

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IMPACT ON RAILWAY BRIDGES

BY

CHARLES T. G. LOONEY
FORMERLY GRADUATE STUDENT IN CIVIL ENGINEERING

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IMPACT ON RAILWAY BRIDGES

I. Introduction

1. *Object and Scope of Investigation.*—The object of this investigation was to examine the literature on the subject of impact on railway bridges; and to study critically certain points of particular interest, with the purpose of contributing to the progress of the work and of indicating ways and means for future investigation. This study consists of five parts:

(1) A summary of the three most important investigations into the subject of impact, namely,

(a) Report of Committee on Impact, Proceedings, American Railway Engineering Association, Volume 12, Part III, 1911;

(b) Report of the Bridge Stress Committee, Department of Scientific and Industrial Research, London, 1928;

(c) Impact on Steel Railway Bridges of Simple Span, J. B. Hunley, Proceedings, American Railway Engineering Association, Volume 38, 1936.

(2) An appraisal of the work which has been done and a statement of the most important work which remains to be done, together with a statement of the portion of the latter which is contained in this investigation.

(3) An explanation of the method of analysis used in this investigation.

(4) An investigation by means of analysis and experimentation into such questions as

(a) the use of the sine curve for the deflection curve of a bridge in the analysis for center oscillations;

(b) the representation of a locomotive by a single axle load in the analysis of bridge oscillations;

(c) the use of the ordinary theory for the forced vibration of a spring to compute the amplitude of oscillations due to hammer blow;

(d) the nature of the dynamic effect of a series of smoothly-rolling loads, called speed effect;

(e) the effect of the phase angle of the hammer blow upon the oscillations;

(f) the dynamic effect of a series of wheels with hammer blow and damping force.

(5) A discussion of the work which remains to be done, together with methods of procedure indicated by the results of this investigation.

2. *Acknowledgments.*—This bulletin is an outgrowth of a thesis presented for the degree of Master of Science at the University of Illinois, and supervised by Professor Hardy Cross; and a thesis presented for the degree of Doctor of Philosophy in Engineering at the same university, and supervised by Professor W. M. Wilson. The step-by-step analysis used in this bulletin was described to the author by Professor Hardy Cross. The development of this method of analysis and the experimental work were done under the direction of Professor W. M. Wilson. The author also owes to Professor Wilson thanks for very considerable help in accomplishing the transition from thesis to bulletin.

The experimental work was done at the University of Iowa. Dean F. M. Dawson and Professor B. J. Lambert of the College of Engineering, University of Iowa, both gave assistance in making this arrangement.

The author also acknowledges the advantage of association with members of the Special Committee on Impact of the American Railway Engineering Association of which J. B. Hunley is chairman.

Several courses in vibrations given at the University of Illinois by Professor H. M. Westergaard, now of Harvard University, provided pertinent information.

Professor L. A. Ware of the University of Iowa aided the author in the design of the electrical apparatus used.

Mr. George Sentinella, instrument maker at the University of Iowa deserves credit for the excellent performance of the experimental apparatus.

Mr. Robert Sykes and Mr. Sol London performed the major portion of the computations and drawings.

II. Report of the American Railway Engineering Association Sub-Committee on Impact, 1911*

3. *General Description of Tests.*—The report of the American Railway Engineering Association's Sub-Committee on Impact, 1911, contained an excellent discussion of the various factors contributing to impact. It described a series of field tests extending over a period of four years, and presented the results with a recommended impact allowance.

*"Report of Sub-Committee on Impact," Proceedings, A.R.E.A., Vol. 12, Part III, 1911.

Forty-five bridges of the types and spans shown below were tested.

Plate Girder Bridges	Number Tested
Under 50 feet	7
50 to 75 feet	8
75 to 100 feet	6

Truss Bridges	Number Tested
Under 100 feet	1
100 to 150 feet	12
150 to 200 feet	6
200 to 250 feet	3
Over 250 feet	2

The truss bridges were for the most part open-floor, through, pin-connected, single-track spans. Most of the plate girders were open-floor, deck, single-track spans. The information given for each bridge included the general dimensions, the make-up of the members and the design live load.

The test trains consisted of a locomotive and a few cars, usually enough to cover the bridge. Complete information was given on the locomotives and cars used in the tests. There were forty different types of locomotives with a wide range of weights and hammer blows. The locomotive-plus-tender weights varied from 230 000 to 420 000 pounds, with the majority over 300 000 pounds. The total hammer blow at 5 r.p.s. varied from 0 up to 80 000 pounds. Several balanced compound locomotives and two electric locomotives were also used; neither of these types had hammer blow.

In the field, records were taken of the center deflection of the bridges at slow velocities of approximately 10 m.p.h.; at this velocity impact effect is negligible. These were called crawl deflections, and were considered to represent the static deflections. Crawl-deflection records were made immediately before and after taking on coal and water so that the weight for any series of test runs could be obtained by interpolation. Records for dynamic effect were made at a number of gradually increasing velocities up to the maximum obtainable, which was usually about 60 to 70 m.p.h.; however, test runs were made at as high as 100 m.p.h. in some instances. Records of the deformation in the various types of members of a bridge were made simultaneously with the deflection records. Each of the records showed a base line, a wavy line of center deflection, or strain, and a scale line drawn on the record by hand. The length of this scale line was equal to the maximum static deflection or strain which was obtained experimentally from the

crawl record. The velocity of the train was taken by observers with stop watches, stationed on a measured base line. The report states, "It was the purpose to secure data, so far as possible, that would give the desired information by direct comparison, only, and without depending on calculations."

4. *Results of Tests.*—The chief causes of impact enumerated in this report were:

"(1) Unbalanced locomotive drivers
(2) Rough and uneven track
(3) Flat or irregular wheels
(4) Eccentric wheels
(5) Rapidity of application of load
(6) Deflection of beams and stringers, giving rise to variations in the action of the vertical load."*

The report stated that impact is caused primarily by the first of these factors. The rotating weights on the drivers cause a periodic force which tends to produce cumulative oscillations.

Sample records were given in the report and the results of all the test runs were tabulated. These tabulated results for each run showed the velocity in miles per hour of the locomotive, the value of the maximum ordinate for deflection or strain in hundredths of inches, the value of the crawl ordinate corrected for coal and water loss, and the percentage impact. The percentage impact was computed by subtracting the crawl ordinate from the maximum ordinate for any run, and dividing by the crawl ordinate. Diagrams were given in the report for each bridge and locomotive tested, showing the impact percentages obtained from deflections as ordinates and velocities as abscissas. These percentages, of course, contain all the other impact effects as well as that due to hammer blow.

5. *Determination of Recommended Impact Allowance.*—The recommended impact allowance was based on the percentage increase of a fast run over a crawl run for center deflections and for the elongation or compression of the chord and the web members as indicated by the extensometer records. It was concluded that the strain measurements for the main truss members and for the flanges of the plate girders were representative of the general impact effect in all the members, and hence provided a simple means of expressing the impact allowance. The values of the maximum percentage increase of the strain for these

*See footnote, page 8.

members, based on the crawl-strain measurements, were plotted as ordinates with the spans of the bridges as abscissas. The curve that enveloped all these values was expressed as,

$$I = \frac{100}{1 + \frac{L^2}{20\,000}}$$

in which L is the loaded span length and I is the percentage of the live-load stress to be added for impact. The curve of this equation is compared with other impact curves in Fig. 1, page 15.

III. Report of the Bridge Stress Committee, Great Britain, 1928*

6. *General Description of Tests.*—The report of the Bridge Stress Committee of Great Britain consisted of a description of extensive field tests, and a complete exposition of a new impact recommendation, together with a description of the method by which it was obtained. In addition, extensive theoretical investigations were made under the direction of C. E. Inglis of Cambridge University.

The report listed the following items under "Nature of the Problem" to be studied by the committee:

"(1) To establish a relation between the pulsating forces which are exerted by a locomotive when traveling at speed, and the consequent oscillations and stresses in the bridge.

(2) To measure any stresses and deflections that are due to the dynamic action of moving loads, unconnected with the hammer blow of locomotives.

(3) To express the probable impact effect on bridges in a form suitable for general use by designers, either by means of a formula, or otherwise.

(4) To investigate the characteristics of locomotives and the possibility of modifying their design, with a view to reducing impact effects."*

Under the title "Object of the Tests" the report listed the following:

"(1) Comparison of the deflection records made on each bridge when the same engine was run at different speeds, in order to find at what frequency maximum oscillation was produced, and whether or not this

*"Report of the Bridge Stress Committee." Department of Scientific and Industrial Research of England. Published by His Majesty's Stationery Office, 1928.

oscillation was due to some degree of synchronism. Sets of runs, at a graded series of speeds, were made on each bridge with various engines.

(2) Comparison of records made on each bridge by different locomotives at the same frequency, in order to see whether the effects produced are proportional to the pulsating forces exerted by the respective locomotives.

(3) Comparison of the records so produced with the curves predicted by theory. This allowed some estimate to be formed of the damping coefficients and other factors which affected the observed results.

(4) Comparison of stress records obtained in various members of the bridge with the records of central deflection, in order to ascertain how far the increase of stress due to vibration is proportional to the increase of the deflection.

(5) Measurement of deflection in the girder and of stress in its chief members under a known distributed load, to enable the impact effect measured by stress or deflection to be expressed as an equivalent distributed load on the bridge.

(6) Examination of the effect of special features in the design of the bridge and of track conditions on the recorded stresses and deflections."*

Fifty-two bridges of the spans listed below were tested:

Plate Girder Bridges	Number Tested
Under 50 feet	19
50 to 75 feet	11
75 to 100 feet	4

Truss Bridges	Number Tested
Under 100 feet	1
100 to 150 feet	8
150 to 200 feet	3
200 to 250 feet	2
Over 250 feet	4

Complete information, which includes a small plan, elevation, and section, length, depth, total weight, and the loaded and unloaded frequency was given for these bridges. The loaded frequency was measured with a locomotive weighing 107 tons placed at the center.

Nearly all were double-track bridges. There was a great variety in the web systems and sway bracing of the trusses. One of the trusses had a lattice web, several had diagonals over two panels similar to

*See footnote on page 11.

Whipple trusses. The shorter spans had the usual Warren and Pratt web systems. It appears from the sketches that it was not common practice to use lateral bracing in the plane of the floor.

Complete information was given on the locomotives used in these tests. This included the axle loading and spacing, hammer blow of the whole locomotive and for each axle, with phase angle. A number of the locomotives had inside cranks. The locomotive-plus-tender weights used in these tests varied from 150 000 to 350 000 pounds, with the majority about 220 000 pounds. The hammer blow for the whole engine varied up to 40 000 pounds at 5 r.p.s. The engines were chosen with special regard to the relation between hammer blow and engine weight. Either one or two engines formed a test train without any cars following. The report stated that the cars had a damping effect on the oscillations. In addition to the information on the locomotives used in the tests, the report gave general information on all types of locomotives in use at that time (1928) in Great Britain.

Tests performed on each bridge were as follows:

(1) Calibration tests to obtain the relationship between load and center deflection. For long bridges one or two engines coupled were used.

(2) Frequency measurements were made with an oscillator on the bridge unloaded, and also with a locomotive standing in several positions.

(3) Tests were made with the locomotives running over the bridges at velocities varying from 8 m.p.h. to the maximum attainable. Center deflections and the strains in the members were measured for each run.

Test runs were made in the field using trains assigned for the purpose. Records were made of the center deflection of the trusses and girders, and extensometer records were taken of the strains in the girder flanges and in the individual members of the trusses. The records showed a time scale, a wavy line of center deflection, or strain, and a mark made by a trigger arrangement attached to the rail. The trigger mark on the deflection or strain diagram indicated the position of the front wheel of the locomotive on the bridge. By means of a wire attached to the driver opposite the counterweight, which made a mark in a clay trough beside the rail, it was possible to mark on the diagrams the position of the maximum value of the resultant hammer blow of the locomotive. Only a few selected records were shown in the report.

Records were also made of the deflections caused by an oscillator

fastened to the center of the bridge, which caused a hammer blow of variable magnitude and frequency.

7. *Results of Tests.*—The report presented the data in the form of redrawn deflection and strain records. In comparing the deflection records made on each bridge, the report showed sets of deflection curves for six different bridges. Each set of curves was made with the same locomotive crossing the bridge at a number of velocities. There were also a number of diagrams for these bridges with the maximum amplitude of oscillation plotted against the r.p.s. of the drivers.

In order to check the computed and measured deflections for bridges on which cumulative oscillations occurred, the full curves showing computed and measured deflections of a 262-foot single-track bridge were compared.

The report compared the computed and measured center deflections for short spans and floor beams where there were no cumulative oscillations. The examples were floor beams and plate girders up to 73 feet in length.

The computed deflection was also checked by measuring the deflection of an experimental bridge and load. The bridge was a steel bar with a span of 12 feet, and an unloaded frequency of 5.7 cycles per second. The ratio of the weight of the load, a single wheel, to the weight of the bridge was 0.4. The magnitude of the hammer blow was not given. One experimental deflection curve was shown with the computed deflection superimposed.

A table of damping coefficients was given in the report. There were three sets of values given, one for high-, one for medium-, and one for low-frequency bridges varying with the span length. The report stated that these values were determined from the rate at which the bridge oscillations died out after the passage of a locomotive. Presumably the ordinary theory for a simple spring with velocity damping was employed.

The conclusion of the report was that the deflection records might represent the impact effect on the chords of bridges over 100 feet in span length.

The report gave the results of tests made on five bridges to determine the distribution of impact stresses in the web as compared with those in the chord. One, for which nearly all the web members on one half of the bridge were tested, was a 210-foot Whipple truss. For the other four, the stress was measured in only one or two web members of each bridge.

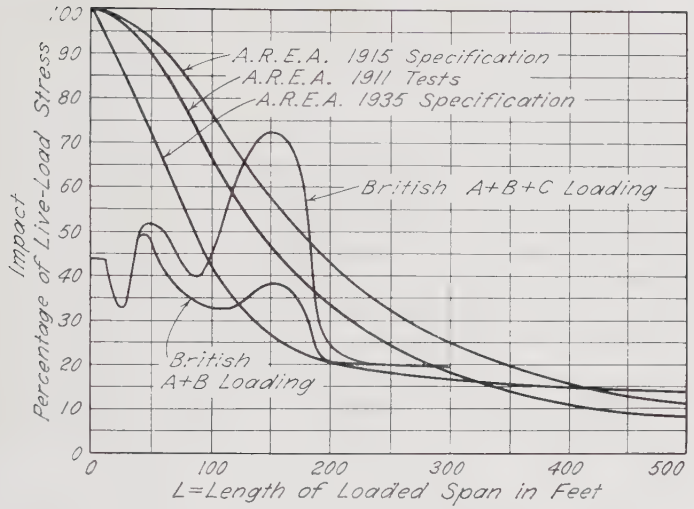


Fig. 1. Various Impact Allowances for Railway Bridges

8. *Determination of Recommended Impact Allowance.*—The experimental data of the Bridge Stress Committee was similar to that of the A.R.E.A. 1911 report, although a different use was made of it in arriving at an impact allowance. The field observations were made to substantiate the theoretical derivation of the recommended impact allowance. The allowance was expressed in tables of equivalent uniform load (live load plus impact) for shear and for moment. The values in the tables were calculated from the theory developed by C. E. Inglis, into which certain constants, based on experimental data, were introduced. Inglis derived a theoretical expression for the center deflection of a bridge due to the hammer blow of an idealized locomotive fixed in position with its wheels skidding. The maximum value of the center deflection was calculated, and the static uniform load which would cause this deflection at the center was the required uniform load in the table, for live load plus the effect of the unbalanced drivers.

Various combinations of the data were introduced into the theory, and the center deflections were calculated in order to obtain the values given in the tables. Curves were drawn showing the relationship between the uniform loads which would cause these maximum center deflections and the span length. The values in the table were taken from the enveloping curves of all combinations.

Curves derived from these tables are shown on Fig. 1, above.

The A and B locomotive loadings are heavy and medium weights with small hammer blow. The C loading is a light locomotive with large hammer blow.

IV. Impact in Steel Railway Bridges of Simple Span*

9. *General Description of Tests.*—The investigation made by J. B. Hunley consisted of tests made in the field, an analytical investigation, and a recommended impact allowance for design purposes.

Tests were begun in 1931 by the Cleveland, Cincinnati, Chicago, and St. Louis Railway on a number of bridges which were in good condition, but which were limited as to usage when rated by the accepted impact requirements of the time. The work was originally undertaken only in order to determine the effect of the impact on these bridges, fifteen in all, before recommending replacements. Later the tests were extended, and, in combination with an analytical investigation, formed the basis for a new A.R.E.A. impact allowance for design purposes.

The bridges tested in this investigation were as follows:

Plate Girder Bridges	Number Tested
Under 50 feet	2
50 to 75 feet	12
75 to 100 feet	11
125 feet	2

Truss Bridges	Number Tested
Under 100 feet	0
100 to 150 feet	5
150 to 200 feet	2
200 to 250 feet	2
Over 250 feet	2

The bridge data consisted of a line diagram of each bridge tested, with the principal dimensions, chord sections for trusses, and flange and web sections for plate girders. The specifications and live loads used in designing the bridges were given. The design live loads varied from E-30 to E-70 so that there was considerable latitude in the dead weights. All but one of the bridges were single track. Of the girder bridges, 19 had open-floors and 8 had concrete decks with ballasted floors. There was a considerable amount of derived data on frequencies and deflections.

*"Impact in Steel Railway Bridges of Simple Span," J. B. Hunley, Proceedings of American Railway Engineering Association, Vol. 37, 1936.

Complete information was given on the locomotives used in these tests. This included the axle spacings and loadings, the frequency of the locomotive springs, details of the drivers and reciprocating parts, the hammer blow per wheel, and the total for the locomotives. For the most part, deflection records were taken during the passage of regularly scheduled passenger and freight trains which consisted of a locomotive-plus-tender and a string of cars. In a few instances a special test train was used. The weights of the locomotive-plus-tender used in these tests varied from 351 000 to 846 000 pounds, and the total hammer blow for all drivers varied from 20 000 to 75 000 pounds at 5 r.p.s.

In the field, records were made of the center deflection of the bridges with a deflectometer. The deflectometer records showed a time scale, a wavy line of center deflection, and a mark indicating the position of the front axle of the locomotive on the bridge in relation to the deflection diagram. The weights of the cars and their contents were obtained from office records, and the weight of the tender was ascertained from the coaling reports.

10. *Results of Tests.*—The causes of impact listed in this report were:

"(1) A periodic alternating force, produced by that portion of the counterweight on the locomotive drivers which has been provided to counterbalance a portion of the reciprocating parts.

(2) Low joints or other irregularities in the track, and eccentric or rough wheels of the locomotives and cars.

(3) Smoothly-rolling loads, such as electric locomotives.

(4) The vertical oscillation of that portion of the car which is carried on the springs.

(5) The rolling or swaying of the locomotive and cars. While the force so produced may, under certain conditions, be periodic in nature, it can well be considered as a static force, and be provided for by an addition to the static live load."*

The report dealt principally with the hammer-blow effect. The results of the field tests were used to obtain working data, including damping coefficients, for use in the theory derived by Inglis, who conducted the analytical work for the Bridge Stress Committee. The theory of Inglis was used by Hunley to compute the recommended impact allowance.

In the field tests made for this report only the center deflections

*See footnote on page 16.

were recorded. Sample records were given and the results of all the runs were tabulated.

The tabulated results for each run gave information concerning the locomotive and tender, the number of cars, the computed static live-load deflection, the loaded frequency, the r.p.s. of the drivers, the maximum semiamplitude of the oscillation, and the percentage impact.

These data provided the information for the drawing of two diagrams for each bridge tested. The first diagram had the maximum semiamplitude of oscillation as ordinates and the r.p.s. of the drivers as abscissas. The second diagram had the dynamic magnifier plotted as ordinates to the r.p.s. of the drivers as abscissas. Curves were drawn through these plotted points, the general shape of the curves being determined by the theory for the forced vibration of a spring with velocity damping. The damping coefficient for the spring curve fitting the plotted points was taken as the damping coefficient for the bridge.

11. *Determination of Recommended Impact Allowance.*—The method used in this report for determining impact allowance for use in designing bridges may be divided into two steps.

(1) The oscillations obtained experimentally with known locomotive and bridge characteristics were used in the analysis to obtain values of the bridge damping factors. The experimental work plus the analysis then furnished values of damping factors for different span lengths.

(2) Designs were made using the A.R.E.A. Specifications for Steel Railway Bridges, with a special locomotive loading and an estimated impact; then the oscillations were computed by the same analysis as in (1), using the derived damping factors.

The designs were revised until the impact assumed in the designs coincided with the impact computed by the analysis. These impact values, derived analytically using the experimental damping factors, formed a basis for the new A.R.E.A. impact allowance.

The theory involved in this investigation* was based on the work of Inglis of Cambridge University, who conducted the analytical investigation for the Bridge Stress Committee. For purposes of analysis, the bridges are divided into three groups:

(1) Short spans—less than 20 to 20 feet in length
(2) Intermediate spans—30 to 175 feet in length, with
    (a) no spring action of the locomotive;
    (b) possible spring action of the locomotive on spans of 100 to 125 feet in length
(3) Long spans—more than 175 feet in length.

*"A Mathematical Treatise on Vibrations in Railway Bridges," C. E. Inglis, Cambridge University Press, London, 1934.

The results of these tests, when taken with the analysis, formed a basis for the impact allowance in the A.R.E.A. Specifications for Steel Railway Bridges which were adopted by the American Railway Engineering Association in March, 1935.

Two formulas, which were the equations of enveloping curves with computed percentages of impact as ordinates to the span lengths as abscissas, were used to express the impact allowance. Each percentage was the quotient of the computed semiamplitude of the oscillation of the center due to hammer blow, and the computed center deflection due to the static live load.

Figure 1, page 15, shows the recommended impact allowance obtained by each of the three reports described.

V. Discussion of Results of Previous Investigations

12. *Introduction.*—It is the purpose of this chapter to summarize and discuss briefly the results of the three investigations described in Chapters II, III, and IV, and to point out some of the problems involved in the study of impact. Certain of these problems have been selected for critical investigation, and a discussion of the methods used in this bulletin, together with a statement of the results obtained, comprise the remaining chapters.

13. *Experimental Work.*—The differences which existed between the types of bridges and locomotives tested in the United States and in England must be taken into account. The principal variations in trusses were found in the lateral and sway-bracing systems, and in the web systems for long-span bridges. Most of the American bridges tested were single-track spans, while the greater number of the British tests were made on double-track bridges. Of the plate-girder bridges tested, the American bridges were usually of the deck type, while those used for the British tests were through girders having a wide variety of floor systems. In general, American locomotives were heavier and had a larger hammer blow than the British locomotives.

In studying the results of these tests it is necessary to consider the fact that there were several independent variables involved in the data for bridge and locomotive. These variables included

(1) Weight of locomotive, and axle spacing
(2) Springing of locomotive
(3) Hammer blow of locomotive
(4) Frequency of hammer blow

(5) Velocity of locomotive
(6) Length of bridge
(7) Weight of bridge
(8) Stiffness of bridge
(9) Amount of damping.

In the A.R.E.A. 1911 report, each of the test trains consisted of a locomotive and tender, with enough cars following to cover the span. In the A.R.E.A. 1936 report, the test trains were, for the most part, regularly scheduled passenger and freight trains. In the British 1928 report, a locomotive and tender alone were used, and it was stated that cars following tended to decrease the oscillation.

Center deflections were measured in the field tests of all three investigations. Strain measurements were made by the A.R.E.A. 1911 committee, and by the British 1928 committee. The test data obtained in the three investigations were approximately the same, but a different use was made of them in each case, in arriving at an impact allowance. Different methods were used by the three investigations to compute impact percentages from the deflection and strain records:

(1) A.R.E.A. 1911 report. The impact percentage was computed from the difference between the maximum total deflection or strain ordinates at high velocity and the crawl ordinates.

(2) A.R.E.A. 1936 report. The impact percentage was computed from the maximum semiamplitude of oscillation for speeds near or below the loaded frequency. For higher speeds the oscillation was taken between the points of maximum oscillation and maximum deflection. The crawl deflection was taken from an average curve derived from a fast run record.

(3) British 1928 report. The impact percentage was computed from the maximum semiamplitude of oscillation.

There are three quantities in the foregoing impact percentage computations; they are

(1) Maximum static deflection
(2) Maximum total deflection at high velocity
(3) Maximum semiamplitude of oscillation.

All three of these may occur for the same position of the locomotive on the bridge, and again it is possible for them to occur for three different positions of the locomotive. Therefore it is possible to compute different impact percentages from the same data. Perhaps different percentages should be computed from the same record. The difference between the maximum static and total deflection might be used for the chord members, while the maximum semiamplitude might be

TABLE 1
EXPERIMENTAL DATA AVAILABLE ON IMPACT

Cause of Impact	A.R.E.A. 1911	A.R.E.A. 1936	Bridge Stress Committee 1928
Unbalanced drivers	Yes	Yes	Yes
Smoothly-rolling load	Yes	No	No
Joints, rough and uneven track	Qualitative	No	One bridge peculiar design
Flat, irregular or eccentric wheels	Qualitative	A little	No
Lurching or nosing	No	Yes	A little
Deflection of beams and stringers giving rise to variations of the action of the load	Qualitative	No	No
Effect of cars	Yes	Yes	No

used for the web members, when the locomotive is near the end of the bridge.

All three reports are concerned chiefly with the hammer-blow effect. Table 1, above, summarizes the data available on the factors which contribute to impact.

14. *Impact Allowance.*—The impact allowance in the three reports was given as an addition to the static live-load stress. Figure 1, page 15, shows the three recommendations plotted on the same diagram. The impact allowance of the A.R.E.A. 1911 report was based entirely on the experimental impact percentages. The curve is an envelope of plotted experimental percentages.

In the British 1928 report, and in the A.R.E.A. 1936 report, the impact allowances were based on theoretically-derived oscillations. The experimental field tests were designed to substantiate the theory and to provide working data. The large impact allowance specified in the British 1928 report for spans approximately 150 feet in length was provided to cover the factor known as spring action of the locomotive, which was introduced in this report.

15. *Theoretical Development.*—Some theoretical derivations were given in the A.R.E.A. 1911 report. The theory dealt with the computation of loaded frequencies, with the effect of a smoothly-rolling load, and with the oscillations due to the hammer blow.

The theory on which the impact allowances of the British 1928 report and of the A.R.E.A. 1936 report were based was developed by Inglis of Cambridge University, and was used to obtain the center oscillation of a bridge due to hammer blow. The use of theoretical methods to obtain the impact allowance is desirable because it makes

possible the isolation of the effect of such variables as the weight of the locomotive, the hammer blow, the weight of the bridge and the frequency of the bridge when studying their individual effects.

It was assumed in the theory of Inglis that the bridge was a simply supported beam of uniform cross-section and the locomotive a single axle load with hammer blow, moving across the beam at a constant velocity. Damping was assumed to be a distributed force resisting motion, and having a magnitude at each point proportional to the vertical velocity of the beam at that point. The deflection of the beam was further assumed to be due to bending only.

The equation representing the vertical motion of the center of the beam was obtained. This was a fourth-order, first-degree differential equation with variable coefficients. The solution was obtained by assuming that the deflection curve was a sine curve defined by the equation

$$y_B = f(t) \sin \frac{\pi x_B}{L} *$$

and by the use of a series solution.

The theory of Inglis was checked in the British 1928 report by comparing the center oscillations computed theoretically with the center oscillations obtained experimentally. The tests were made on a 262 foot 6 inch Whipple truss whose total weight was 1 030 000 pounds, and on a locomotive and tender with a wheel base of 44 feet 7½ inches and a total weight of 241 000 pounds. The theory was also checked experimentally in the laboratory by means of a single-axle load with hammer blow which passed over a steel bar. The bar was 12 feet long, and had an unloaded frequency of 5.7 cycles per second. The weight of the load and the hammer blow were not given. The ratio of the weight of the load to the weight of the bar was 0.4.

The foregoing theory was not used for the purpose of computing the impact allowance because of its complexity. Instead, a modified theory was used, which was based upon the assumption that the hammer blow was caused by a single axle load stationary at the center of the bridge with its wheels slipping. The action of this system is similar to that for a simple spring with a forced vibration and velocity damping. It will be noted that the frequency of this system is constant; that the number of applications of the hammer blow is unlimited; and that the amplitude of the oscillation is restricted by damping. The British 1928 report stated that the two theories, i.e., that concerned

*See notation, Section 18.

with a moving load and the modified theory dealing with a stationary load, were in close agreement, and that the latter erred on the side of safety.

16. *Problems to Be Studied in Present Investigation.*—An examination of the reports of the previous investigations on impact revealed certain problems which warranted further critical study. An experimental check was made in the field in the British 1928 report to substantiate the theory that a single axle load can represent the locomotive. In the field test the locomotive had a wheel base of approximately ⅙ of the span length of the bridge. It is reasonable to question the accuracy of this representation when it is applied to conditions in the United States, where the locomotive wheel base may be 83 feet and the span length of the bridge 75 feet.

In the theory developed by Inglis for the British 1928 report it was assumed that the deflection curve of the beam is a sine curve having the equation,

$$y_B = f(t) \sin \frac{\pi x_B}{L}^* .$$

This sine curve relationship for the deflections also establishes a sine curve relationship between the accelerations at different points on the beam. This latter relationship is important in computing the effect on the oscillation of the mass of a load on the bridge. This assumption may profitably be checked by means of laboratory tests.

Another problem is presented by the limitation of the number of applications of the hammer blow on short spans. This limitation must be reconciled with the modified theory of Inglis for a stationary load which assumes an unlimited number of applications of the hammer blow, and a damping restriction.

In the British 1928 report the experimental checks of the theory of Inglis for a moving load, made in the field and in the laboratory, used a load which was small in comparison with the weight of the bridge. The load reduced the frequency of the bridge in the field test by 16 per cent, and in the laboratory by 25 per cent. A question arises in this connection as to the accuracy of the modified theory of Inglis which assumes a constant frequency, when the theory is applied to locomotives and bridges in the United States where the locomotives are heavier relative to the bridges than they are in England. A bridge of medium length and a locomotive, selected from the A.R.E.A. 1936

*See notation, Section 18.

report, weighed, respectively, 136 000 and 650 000 pounds. The change of the bridge frequency due to the weight of the locomotive for this combination was from 50 to 60 per cent, as compared with the 16 to 25 per cent change reported in the test previously mentioned.

It is possible that the two last-mentioned factors, i.e., the small limited number of applications of the hammer blow, and the variation in frequency caused by the weight of the locomotive, may be important for bridges of medium length. The limitation of the amplitude of oscillation which these factors cause may have been included under the term "damping" in previous reports.

One of the first requirements for the further development of the problem of impact is the assembling of data on the damping of bridge oscillations. Before this question can be approached intelligently, it will be necessary to distinguish the difference between true damping, considered as a loss of energy through friction and inelasticity, and the other factors that cause the observed values of oscillation to be less than the values based upon the theory of a spring with forced vibration. It would be highly desirable to have information on the individual effects of the various independent factors listed on pages 19 and 20. This information would aid in designing field tests and interpreting the data.

In order to examine some of the questions discussed in the foregoing, a new method of analyzing railway bridges has been developed for the present investigation which will determine the effect of each of several factors, and which will accommodate a series of axles having the spacings and weights of the locomotive axles. The analysis was supplemented by tests made in the laboratory. Apparatus was designed and built which made possible the measurement of the oscillations caused by a single load or by a series of loads, with and without hammer blow, and with or without damping. Chapters VI through X of this bulletin contain a detailed exposition of the analysis and method of experimentation which, it is believed, will make this possible.

VI. Step-by-Step Analysis Applied to Railway Bridges

17. *Introduction.*—The purpose of the analysis is to determine the stresses produced in a railway bridge by the passage of a locomotive. These stresses are due in part to the weight of the structure and of the locomotive and in part to the impact resulting from the movement of the locomotive and bridge. For a given distribution of load, the stresses at the center of a bridge are proportional to the deflection.

Moderate variations in the load distribution do not greatly affect the relation between the stress and the deflection at the center of the span, and the present analysis is based upon the assumption that the central deflection is a measure of the stress. The analysis is thus a problem of the kinetics and kinematics of the motions incident to the passage of a locomotive over a bridge.

It is not the purpose of this investigation to reproduce all the true field conditions, but rather to separate and control the different factors. The factors that enter into one field test are numerous, and occur simultaneously. It is the purpose of this analysis and of the laboratory tests to idealize certain factors, and to eliminate some and combine others, to the end that the effect of each factor may be more fully understood and evaluated.

The problem which arises from the passage of a locomotive over a bridge is considered to be one of plane motion in which the bodies, forces, and movements are in one plane. The locomotive is considered to be a series of independent wheels with the same spacings and loadings as the locomotive axles. The hammer blow is considered to be a vertical sinusoidal force which is associated with the wheels representing the drivers of the locomotive. The total hammer blow is equally divided among the drivers so that the forces considered are equal in magnitude and in phase. The bridge is assumed to be a simply-supported beam of uniform cross-section, i.e., uniform distribution of mass and stiffness. The deflection of the bridge is considered as due to bending only, so that the rate of change of slope is proportional to the bending moment.

The step-by-step analysis, which is described in this bulletin, is a numerical solution of a differential equation in which the movement of the bodies is imagined to occur in small finite steps or displacements corresponding to a series of very small equal finite intervals of time. During each small displacement the acceleration and velocity are considered to be linear functions of time. At the end of each small displacement a condition of equilibrium is established between the bodies and forces by the use of D'Alembert's Principle. The equations which result from these conditions make it possible to compute the small increments to the acceleration, velocity, and displacement of the bodies for each small interval of time, and the continuous functions of acceleration, velocity, and deflection may be constructed with relation to time, beginning with any known state of motion.

18. *Assumptions and Notations.*—Before developing the equation which is fundamental to the application of the step-by-step analysis

to railway bridges, the nature of the problem and the assumptions involved in its solution will be summarized for the convenience of the reader. They are as follows:

(1) The analysis deals with the dynamic and static effect of the locomotive upon the deflection of the bridge.

(2) The bridge is considered to be a simply-supported beam of uniform mass and moment of inertia, and the deflection is considered to be due to bending only.

(3) The deflection of the center of the bridge is considered to be a measure of the stress in the bridge.

(4) The vertical deflection, velocity, and acceleration of the center of the bridge are found for all positions of the locomotive, the front driver of the locomotive being used as a reference point for the position of the locomotive relative to the left end of the bridge.

(5) The locomotive is considered to be a series of independent wheels with loads and spacings equal to the locomotive static axle loads and spacings. The entire weight is considered unsprung.

(6) In the computation of the center deflection due to the static dead, live load, and hammer-blow forces the usual beam theory is used. In order to obtain the relation between deflection, velocity, and acceleration, the deflection curve of the bridge is assumed to be a sine curve.

The following notation has been used:

a = center acceleration of bridge
C = circumference of driver
d = total center deflection of bridge
Δ = damping coefficient of bridge used by Hunley
$\Delta_{D.L.}$ = center deflection of bridge due to dead load
$\Delta_{H.B.}$ = center deflection of bridge due to hammer blow considered as static force
$\Delta_{L.L.}$ = center deflection of bridge due to a live load considered as static force
$\Delta_{M.B.}$ = effect on center deflection of bridge of acceleration of mass of bridge
$\Delta_{M.L.}$ = effect on center deflection of bridge of acceleration of mass of locomotive
F = force between wheel and rail
g = acceleration of gravity
$H.B.$ = maximum hammer blow of any driver axle at a given velocity
K = modulus of bridge; the number of pounds at the center necessary to make the center deflect one inch

L = span length of bridge
m = mass per foot of bridge
n_b = damping coefficient of unloaded bridge; $4\pi n_b m$ = damping force per foot of bridge, per foot per second of vertical velocity. This coefficient was used by Inglis.
n_b' = damping coefficient of loaded bridge
n_0 = frequency of unloaded bridge
n_2 or n_0' = frequency of loaded bridge
m.p.h. = miles per hour
P = any axle load of locomotive
R = ratio between successive oscillations of a damped free vibration
r.p.s. = revolutions per second
Δt = small time interval
$f(t)$ = function of time which determines d, the center deflection
v = center velocity of bridge
w = weight of bridge per foot, assumed to be uniformly distributed
x_B = horizontal distance of any point B from left end of bridge
x_P = horizontal distance of any wheel P from left end of bridge
$\dot{x}_P$ = constant horizontal component of velocity of any wheel P
y_B = deflection of any point B a distance x_B from left end of bridge
$\dot{y}_B$ = velocity of any point B a distance x_B from left end of bridge
$\ddot{y}_B$ = acceleration of any point of bridge a distance x_B from left end of bridge
y_P = deflection of any wheel P at a distance x_P from left end of bridge
$\dot{y}_P$ = vertical component of velocity of any wheel P at a distance x_P from left end of bridge
$\ddot{y}_P$ = vertical acceleration of any wheel P at a distance x_P from left end of bridge
$\cong$ = equals approximately

The development of the fundamental equation follows and, subsequently, an application of the analysis to a bridge and locomotive. The development of the fundamental equation requires the kinematic relations for the bridge and locomotive axles, an expression for the deflection of the center of the bridge, and an equation for the dynamic equilibrium of the bridge and locomotive in any position.

19. *Kinematics of Bridge.*—The assumption that the deflection curve of the bridge is a sine curve of the form $y_B = d \sin \frac{\pi x_B}{L}$, makes it possible to write a very simple expression for the motion of the bridge. The motion of the center of the bridge will be selected as a parameter and, with the assumed sine curve, the motion of every other point of the bridge is fixed.

The deflection of a bridge is very small relative to its length, and grades for railway bridges are very low, seldom over 0.1 per cent, so that the motion of any point of the beam is essentially vertical. The origin of the coordinate axes is taken at the top of the rail at the left end of the bridge and the axis, x, is passed through the top of the rail at the right end, even though the bridge is on a grade.

The following equations define the vertical motion of the center of the bridge.

$$\text{Center deflection} \quad d = \Delta_{\text{D.L.}} + f(t) \tag{1a}$$

$$\text{Center velocity} \quad v = f'(t) \tag{1b}$$

$$\text{Center acceleration} \quad a = f''(t). \tag{1c}$$

The corresponding equations for any other point on the bridge are as follows:

$$y_B = [\Delta_{\text{D.L.}} + f(t)] \sin \frac{\pi x_B}{L} = d \sin \frac{\pi x_B}{L} \tag{2a}$$

$$\dot{y}_B = f'(t) \sin \frac{\pi x_B}{L} = v \sin \frac{\pi x_B}{L} \tag{2b}$$

$$\ddot{y}_B = f''(t) \sin \frac{\pi x_B}{L} = a \sin \frac{\pi x_B}{L}. \tag{2c}$$

20. *Kinematics of Wheels Representing Locomotive.*—The kinematic relations for the bridge are given by Equations (2a), (2b), and (2c). The corresponding relations for the wheels representing the locomotive have the same root equation, (2a), as the bridge. This follows from the fact that the vertical displacement of any wheel is the same as that of the point of the bridge directly beneath it. The wheels of the locomotive move across the bridge with uniform velocity but, although the vertical displacement of any wheel is the same as that of the point of the bridge directly beneath it, the velocity and accelera-

tion of the two are not equal in either direction or magnitude. The distance x_P to any wheel P from the left end of the bridge is a function of time, whereas the distance x_B of any point of the bridge is not a function of time. This difference makes the derived equations (3b) and (3c) for the wheels differ from the corresponding equations (2b) and (2c) for the bridge. The dynamic relations for the motion of the wheels representing the locomotive are as follows:

$$y_P = [\Delta_{D.L.} + f(t)] \sin \frac{\pi x_P}{L} = d \sin \frac{\pi x_P}{L} \tag{3a}$$

$$\dot{y}_P = v \sin \frac{\pi x_P}{L} + d \frac{\pi}{L} \dot{x}_P \cos \frac{\pi x_P}{L} \tag{3b}$$

$$\ddot{y}_P = a \sin \frac{\pi x_P}{L} + 2v \frac{\pi}{L} \dot{x}_P \cos \frac{\pi x_P}{L} - d \frac{\pi^2}{L^2} (\dot{x}_P)^2 \sin \frac{\pi x_P}{L}. \tag{3c}$$

21. *Deflection of Center of Bridge.*—The stress at the center of a bridge due to a given load distribution is proportional to the center deflection of the bridge. The ratio of the total stress (live load plus impact) to the static live-load stress equals the ratio of the total center deflection (live load plus impact) to the static live-load center deflection. This being true, a knowledge of the vertical movement of the center of a bridge during the passage of a wheel or a series of wheels makes it possible to determine the impact stress in terms of the static live-load stress. The method by which the vertical movement can be computed is described in the following paragraphs.

At any time t', let the deflection of the center of the bridge be d', the velocity v', and the acceleration a'; and at time t'', a small interval of time, Δt, later, let the center deflection be d'', the velocity v'' and the acceleration a''.

Referring to Figs. 2a and 2b, page 30, which represent the velocity–time and the acceleration time curves for the center of the bridge, the area under the velocity–time curve is, by definition, equal to the change of displacement.

Change of displacement:*

$$d'' - d' \cong v' \Delta t + (v'' - v') \frac{\Delta t}{2}. \tag{4}$$

*There will be an error equal to the area between the curve and the straight-line chord, see Fig. 2, page 30.

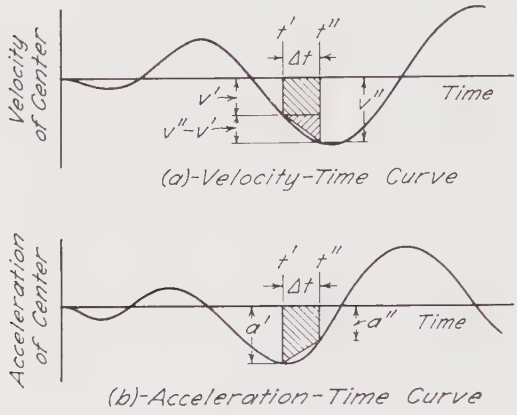


FIG. 2. VELOCITY-TIME AND ACCELERATION-TIME CURVES FOR CENTER OF BRIDGE

The area under the acceleration–time curve is, by definition, equal to the change of velocity.

Change of velocity:*

$$v'' - v' \cong a' \frac{\Delta t}{2} + a'' \frac{\Delta t}{2}. \tag{5}$$

Combining Equations (4) and (5):†

$$d'' = d' + v'\Delta t + a' \left(\frac{\Delta t}{2}\right)^2 + a'' \left(\frac{\Delta t}{2}\right)^2. \tag{6}$$

The deflection of the center of the bridge at any time can be obtained by means of an equilibrium equation for the bridge expressed in terms of the loads (including the accelerative forces as determined by D'Alembert's Principle) and of the elastic properties of the bridge. This deflection may be separated into five factors, as follows:

(1) Dead-load deflection ..... $\Delta_{D.L.}$
(2) Static live-load deflection ..... $\Delta_{L.L.}$
(3) Static hammer-blow deflection ..... $\Delta_{H.B.}$
(4) Deflection due to the acceleration of the mass of the bridge ..... $\Delta_{M.B.}$
(5) Deflection due to the acceleration of the mass of the locomotive ..... $\Delta_{M.L.}$

The center deflection due to each of these factors is determined in the following paragraphs.

*There will be an error equal to the area between the curve and the straight-line chord, see Fig. 2, above.

†The discrepancy has been noted, and Δt will be kept small in order to reduce the error to a negligible quantity.

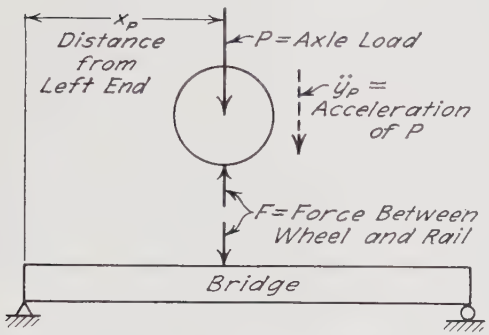


FIG. 3. FORCE BETWEEN LOCOMOTIVE WHEEL AND RAIL

(1) The dead-load deflection, $\Delta_{D.L.}$, is the actual deflection of the center of the bridge from a straight line tangent to the top of the rail at the ends. It includes the deflection due to the weight of the structure, the effect of camber, and the change in elevation of the rail due to leveling of the track. To be certain of this quantity for a bridge subjected to a field test, it would be desirable to make field measurements.

(2) The static live-load deflection, $\Delta_{L.L.}$, is found by the usual theory for the deflection of a beam due to bending. As indicated in the preliminary assumptions, the locomotive is considered to be a series of independent wheels with loads and spacings equal to the locomotive static axle loads and spacings.

(3) The static hammer-blow deflection, $\Delta_{H.B.}$, is found by the usual theory for the deflection of a beam due to bending. The hammer-blow force is considered to vary in a sinusoidal manner. The total hammer blow of the locomotive is considered to be equally divided among the driving axles and to be in phase.

(4) The deflection of the center of the bridge due to the acceleration of the mass of the locomotive may be found in the following manner. It is assumed in the present analysis that the friction of the springs is great enough to cause the whole locomotive to move vertically with the axles, and the total mass of the locomotive is distributed among the axles in proportion to the axle loads. If the spring friction is not great enough to cause the locomotive to move vertically with the axles, a distinction must be made between the sprung and unsprung weights.

Any vertical acceleration of the locomotive will influence the force of the wheels on the rail. Consider an axle, with a proportional share of the weight of the locomotive, as a free body, as shown in Fig. 3.

Then since force equals mass times acceleration,

$$P - F = \frac{P}{g}\ddot{y}_P, \quad \text{or} \quad F = P - \frac{P}{g}\ddot{y}_P .$$

The force of the wheel on the rail, F, is divided into two parts. The effect of P as a static force is included under $\Delta_{L.L.}$, (2) in the foregoing, and the effect of the acceleration of the mass of the locomotive is equal to a force of $-\frac{P}{g}\ddot{y}_P$ for each wheel.

The center deflection due to this force for each wheel is obtained by means of Maxwell's Theorem of Reciprocal Deflections, which indicates that the deflection at the center due to a load at a distance, x, from the left end of the bridge, is equal to the deflection at the distance, x, from the end due to the load at the center. Taking the load as $-\frac{P}{g}\ddot{y}_P$, then the deflection at the center due to the load at a point a distance, x_P, from the left end of the bridge would be

$$\left(-\frac{P}{K}\frac{\ddot{y}_P}{g}\sin\frac{\pi x_P}{L}\right).$$

In accordance with the foregoing analysis, the effect of the acceleration of the mass of the locomotive on the center deflection will be given by the equation

$$\Delta_{M.L.} = -\Sigma\frac{P}{K}\frac{\ddot{y}_P}{g}\sin\frac{\pi x_P}{L} .$$

(5) The effect of the vertical acceleration of the mass of the bridge upon its center deflection is equivalent to the effect of a distributed force in a direction opposite to the acceleration. Since the mass per foot of the bridge has been assumed to be uniform, this distributed load at any point on the bridge will be proportional to the acceleration of that point.

The acceleration of the bridge at any point a distance x_B from the left end given by Equation (2c) is

$$\ddot{y}_B = a\sin\frac{\pi x_B}{L} .$$

The distributed load due to this acceleration is given by the equation

$$(\text{Load per foot}) = -\frac{w}{g} a \sin \frac{\pi x_B}{L}$$

where

$$\frac{w}{g} = \text{the mass per foot of the bridge.}$$

This load will produce a center deflection equal to $-\left(\frac{48wLa}{\pi^4 Kg}\right)$. Since $\frac{48}{\pi^4} \cong \frac{1}{2}$, the effect on the center deflection of the acceleration of the mass of the bridge is given approximately by the equation

$$\Delta_{\text{M.B.}} = -\frac{1}{2}\frac{wL}{K}\frac{a}{g}.$$

To summarize, the factors that compose the center deflection are

(1) $\Delta_{\text{D.L.}}$

(2) $\Delta_{\text{L.L.}}$

(3) $\Delta_{\text{H.B.}}$

(4) $\Delta_{\text{M.L.}} = -\frac{P}{K}\frac{\ddot{y}_P}{g}\sin\frac{\pi x_P}{L}$

(5) $\Delta_{\text{M.B.}} = -\frac{1}{2}\frac{wL}{K}\frac{a}{g}.$

Adding together the five factors, the center deflection, d'', at a particular time, t'', will be given by the equation

$$d'' = \Delta_{\text{D.L.}} + \Delta_{\text{L.L.}}'' + \Delta_{\text{H.B.}}'' - \Sigma \frac{P}{K}\frac{\ddot{y}_P''}{g}\sin\frac{\pi x_P''}{L} - \frac{1}{2}\frac{wL}{K}\frac{a''}{g}. \quad (7)$$

22. *Fundamental Equation of Step-by-Step Analysis.*—The step-by-step analysis is based upon a fundamental equation in which the acceleration of the center of the bridge, a'', at a time, t'', is expressed in terms of the known properties of the bridge and locomotive and of the known acceleration, velocity, and deflection, a', v' and d', respectively, at a time, t', a known interval of time, Δt, before t''. This equation makes it possible to compute from a known state of motion at the time, t', the acceleration, a'', a small interval of time,

Δt, later. With the value of a'' known, v'' and d'' can be computed, and thus a new state of motion is obtained and one step of the step-by-step analysis is completed.

The fundamental equation is obtained by combining Equations (6), (7) and (3c) which were derived in Sections 20 and 21. The equations, repeated here for the convenience of the reader, are as follows:

$$d'' = d' + v'\Delta t + a'\left(\frac{\Delta t}{2}\right)^2 + a''\left(\frac{\Delta t}{2}\right)^2 \tag{6}$$

$$d'' = \Delta_{\text{D.L.}} + \Delta_{\text{L.L.}}'' + \Delta_{\text{H.B.}}'' - \Sigma\frac{P}{K}\frac{\ddot{y}_P''}{g}\sin\frac{\pi x_P''}{L} - \frac{1}{2}\frac{wL}{K}\frac{a''}{g} \tag{7}$$

$$\ddot{y}_P'' = a''\sin\frac{\pi x_P''}{L} + 2v''\frac{\pi}{L}\dot{x}_P\cos\frac{\pi x_P''}{L} - d''\frac{\pi^2}{L^2}(\dot{x}_P)^2\sin\frac{\pi x_P''}{L}. \tag{3c}$$

Substituting the value of $\ddot{y}_P''$ from Equation (3c) in Equation (7), eliminating d'' from Equations (6) and (7), and substituting $v' + a'\frac{\Delta t}{2} + a''\frac{\Delta t}{2}$ for v'', gives

$$\begin{aligned} a''\left[\left(\frac{\Delta t}{2}\right)^2 - \left(\frac{C\pi}{32L}\right)^2 A'' + A'' + \frac{1}{2}\frac{wL}{K}\frac{1}{g} + \left(\frac{C\pi}{16L}\right)B''\right] \\ = \Delta_{\text{D.L.}} + \Delta_{\text{L.L.}}'' + \Delta_{\text{H.B.}}'' - d'\left[1 - \frac{\pi^2}{L^2}(\dot{x}_P)^2 A''\right] \\ - v'\left[\Delta t - \Delta t\frac{\pi^2}{L^2}(\dot{x}_P)^2 A'' + 2\frac{\pi}{L}\dot{x}_P B''\right] \\ - a'\left[\left(\frac{\Delta t}{2}\right)^2 - \left(\frac{C\pi}{32L}\right)^2 A'' + \left(\frac{C\pi}{16L}\right)B''\right] \end{aligned} \tag{8}$$

in which

$$A = \Sigma\frac{P}{K}\frac{1}{g}\sin^2\frac{\pi x_P}{L} = \frac{1}{2}\Sigma\frac{P}{K}\frac{1}{g}\left(1 - \cos 2\frac{\pi x_P}{L}\right) \tag{9}$$

$$B = \Sigma\frac{P}{K}\frac{1}{g}\sin\frac{\pi x_P}{L}\cos\frac{\pi x_P}{L} = \frac{1}{2}\Sigma\frac{P}{K}\frac{1}{g}\sin 2\frac{\pi x_P}{L}. \tag{10}$$

In obtaining Equation (8), the following relations were used,

$$\Delta t = \frac{1}{16}\,\frac{1}{\text{r.p.s.}} \text{ seconds}$$

$$\dot{x}_P = \text{C times r.p.s., feet per second}$$

where,

r.p.s. = the revolutions per second of the drivers
C = circumference of the drivers.

Equation (8) is the fundamental equation of the step-by-step analysis. By choosing Δt as a function of r.p.s., most of the terms in the equation are made independent of the velocity. As a result, most of the terms depend only on the properties of the bridge and locomotive and, once determined, may be used for analyses at different velocities, thus greatly reducing the work.

23. *Method of Making Computations.*—In this analysis the locomotive is moved across the bridge from left to right in a series of equal steps, the length of each step being arbitrarily chosen as $\frac{1}{16}$ of the circumference of the driver (about 1.25 feet). Since Δt is the time interval for one step, then $\Delta t = \frac{1}{16}\cdot\frac{1}{\text{r.p.s.}}$* as previously stated.

If a', v' and d' are known for any position of the locomotive at time t', the acceleration a'', after a time interval Δt, during which the locomotive has moved forward one step, can be computed by the use of Equation (8). With a'' known, values of v'' and d'' can also be found for the new position of the locomotive, by the use of Equations (5) and (6). These values, a'', v'' and d'', then become the a', v' and d' values for the next step.

It will be noted from an inspection of Equation (8) that certain quantities which are functions of the bridge, the locomotive, and the velocity, must be known. Some of these quantities change as the position of the locomotive changes, and therefore must be computed for each step, or at intervals of $\frac{1}{16}$ of the circumference of the driver for the length of the bridge. The successful application of the step-by-step analysis depends upon the use of a systematic method of making the computations. The following system is suggested:

Tabulate the quantities dependent upon the properties of the loco-

*The basis for the selection of this value of Δt is explained on page 61.

motive and the bridge for each step. These are the quantities A and B of Equation (8).

Compute the coefficients of a'', a', v', and d' given in Equation (8) for all steps to be considered in the computations. Also compute the values of $\Delta_{D.L.}$, $\Delta_{L.L.}$, and $\Delta_{H.B.}$, for all positions of the locomotive.

Start the analysis with the first wheel of the locomotive in a position just to the left of the left end of the bridge. In this position $a' = 0$, $v' = 0$, $d' = \Delta_{D.L.}$, and these are the initial values of a, v, and d at the start of the analysis. Then move the locomotive forward one step, bringing the first wheel onto the bridge, and apply Equation (8) to determine a''. With a'' known, compute v'' and d'' for the first step by the use of Equations (5) and (6).

Use the values of a'', v'', and d'' at the end of the first step as the a', v', and d' for the second step, and compute the values of a'', v'', and d'' at the end of the second step by the method outlined above. Repeat the process until the locomotive has been taken across the bridge as far as seems necessary.

The method of analysis will be further illustrated by means of a numerical example. The bridge used in the example is bridge 21 of the report by J. B. Hunley entitled, "Impact on Railway Bridges"; Proceedings, A.R.E.A., Vol. 37, 1936, page 741, and the locomotive is designated as locomotive J1-D of the same report.

Data relative to bridge 21 and locomotive J1-D follow.

Bridge Data

Span length	= 75 feet 6 inches	
Depth of girder	= 8 feet 3¼ inches	
Ratio of length to depth	= 9.13	
Number of tracks	= 1	
		Unit Stresses
Designed for Cooper's E-60	D.L.	1 720 p.s.i.
	L.L.	8 070 p.s.i.
	I	6 450 p.s.i.
		16 240 p.s.i.
Weight per foot of bridge:	steel	1 245 lb.
	deck	510 lb.
		1 755 lb.
Total weight of span		136 000 lb.
Moment of inertia at the center (gross)		501 080 in.^{4}
Moment of inertia at the center (net)		431 700 in.^{4}
Moment of inertia as determined from deflection observations		449 470 in.^{4}

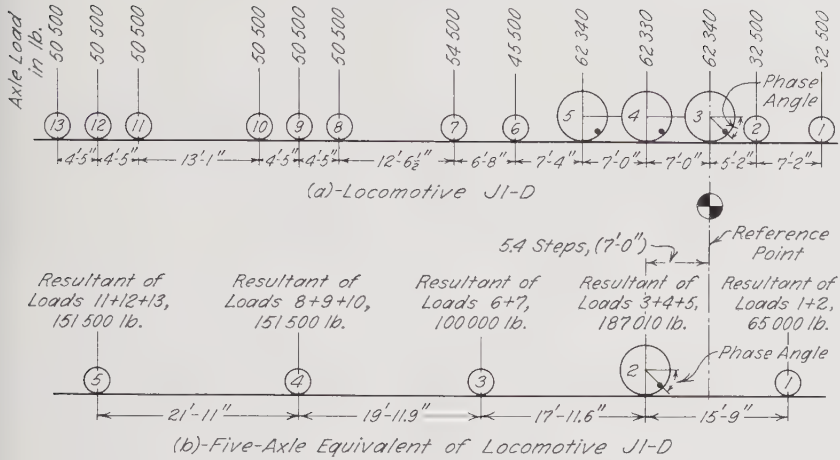


Fig. 4. Axle Loads and Spacings; Locomotive J1-D and Five-Axle Equivalent Locomotive

Locomotive Data

Total weight of engine	= 353 000 lb.
Total weight of tender	= 303 000 lb.
Total weight of engine plus tender	= 656 000 lb.
Total hammer blow of the drivers at 5 r.p.s.	= 42 275 lb.
Circumference of drivers	= 20.7 ft.

The properties of the bridge which enter into the computations are as follows.

Length	L =	75.5 feet
Total weight	wL =	136 000 pounds
Modulus	K =	840 000 lb. per in.
Dead-load deflection	$\Delta_{D.L.}$ =	0.145 inches

The properties of the locomotive that enter into the computations are the axle loads and spacings. They are shown in Fig. 4a, above. Wheels 3, 4, and 5 are the drivers.

Total weight of locomotive	ΣP =	656 000 pounds
Circumference of driver	C =	20.7 feet
Maximum total hammer blow	$\Sigma H.B.$ =	42 275 pounds at 5 r.p.s.

The locomotive is moved across the bridge from left to right. The position of the locomotive on the bridge is determined by the distance of a reference point from the left end of the bridge. The distance is

given in units of $\frac{1}{16}$ of the circumference of the driver, called steps. The reference point for locomotive J1-D is the first driver, wheel 3. In the initial position at the start of the analysis, step (-10), the first wheel of the locomotive is 8 inches to the left of the left end of the bridge. Wheel 3, the first driver of the locomotive and the wheel used as the reference point of the locomotive, is 10 steps to the left of the left end of the bridge. The total hammer blow of the locomotive is divided equally between the three drivers, and the hammer blow of all three drivers is considered to be in phase. The magnitude of the hammer-blow force for any position of the locomotive is determined by the angular position of a counterweight on the driver when the reference point of the locomotive arrives at the left end of the bridge. This angle, called phase angle, is measured clockwise from a horizontal line through the axle of the driver pointing toward the front of the locomotive. The phase angle in this analysis is adjusted to zero degrees so that when the first driver, the reference point, is even with the left end of the bridge, the hammer blow will be zero and increasing in magnitude downward. This is the position for step 0 of the analysis. It is to be noted that the analysis began at step (-10) with wheel 1 slightly to the left of the left end of the bridge.

As outlined in the foregoing, Equations (9) and (10), the values of

$$A = \frac{1}{2} \Sigma \frac{P}{K} \frac{1}{g} \left(1 - \cos 2 \frac{\pi x_P}{L}\right)$$

and,

$$B = \frac{1}{2} \Sigma \frac{P}{K} \frac{1}{g} \sin 2 \frac{\pi x_P}{L}$$

are computed first. This is accomplished in the following manner:

(1) Compute $2 \frac{\pi x_P}{L}$ for each wheel on the bridge for each position of the locomotive as determined by the position of the first driver, for Steps $-10, -9, -8, \ldots, 0, 1, 2, \ldots, 80$.

(2) Obtain the values of $\left(1 - \cos 2 \frac{\pi x_P}{L}\right)$ and $\sin 2 \frac{\pi x_P}{L}$ for each wheel on the bridge for steps $-10, -9, -8, \ldots, 0, 1, 2, \ldots, 80$.

(3) Multiply the values of $\left(1 - \cos 2 \frac{\pi x_P}{L}\right)$ and $\sin 2 \frac{\pi x_P}{L}$ by $\frac{1}{2} \cdot \frac{P}{K} \cdot \frac{1}{g}$ for each wheel for steps $-10, -9, -8, \ldots, 0, 1, 2, 3, \ldots, 80$.

(4) Determine the summation of the values of $\frac{1}{2} \cdot \frac{P}{K} \cdot \frac{1}{g} \left(1 - \cos 2\frac{\pi x_P}{L}\right)$ and $\frac{1}{2} \cdot \frac{P}{K} \cdot \frac{1}{g} \sin 2\frac{\pi x_P}{L}$ for all the wheels that are on the bridge at each step and tabulate

$$\frac{1}{2} \Sigma \frac{P}{K} \frac{1}{g} \left(1 - \cos 2\frac{\pi x_P}{L}\right) = A$$

and,

$$\frac{1}{2} \Sigma \frac{P}{K} \frac{1}{g} \sin 2\frac{\pi x_P}{L} = B$$

for steps —10, —9, —8, . . . , 0, 1, 2, . . . , 80.

Compute the coefficients of a'', a', v', and d' given in Equation (8). These coefficients, repeated for convenience in reference, are

$$\left[\left(\frac{\Delta t}{2}\right)^2 - \left(\frac{C\pi}{32L}\right)^2 A'' + A'' + \frac{1}{2}\frac{wL}{K}\frac{1}{g} + \left(\frac{C\pi}{16L}\right) B''\right] \text{(coefficient of } a'')$$

$$\left[\left(\frac{\Delta t}{2}\right)^2 - \left(\frac{C\pi}{32L}\right)^2 A'' + \left(\frac{C\pi}{16L}\right) B''\right] \quad \text{(coefficient of } a')$$

$$\left[\Delta t - \Delta t \frac{\pi^2}{L^2} (\dot{x}_P)^2 A'' + 2\frac{\pi}{L} \dot{x}_P B''\right] \quad \text{(coefficient of } v')$$

$$\left[1 - \frac{\pi^2}{L^2} (\dot{x}_P)^2 A''\right] \quad \text{(coefficient of } d').$$

The computations for the coefficients of a'' and a' are made in the following manner:

(1) Tabulate the values of A and B for each step

(2) Multiply A by $\left(\frac{C\pi}{32L}\right)^2$

(3) Multiply B by $\left(\frac{C\pi}{16L}\right)$

(4) Subtract (2) from (3), obtaining $\left[-\left(\frac{C\pi}{32L}\right)^2 A + \left(\frac{C\pi}{16L}\right) B\right]$

(5) Add A and $\frac{1}{2} \cdot \frac{wL}{K} \cdot \frac{1}{g}$ to (4), obtaining

$$-\left(\frac{C\pi}{32L}\right)^2 A + A + \frac{1}{2} \cdot \frac{wL}{K} \cdot \frac{1}{g} + \left(\frac{C\pi}{16L}\right) B$$

for each step —10, —9, —8, . . . , 0, 1, 2, 3, . . . , 80. The quantity in (5) will be the same for all velocities of the locomotive used in the analysis. If an analysis is made at 5 r.p.s., Δt will be $\frac{1}{16} \times \frac{1}{5} = \frac{1}{80}$ second and $\left(\frac{\Delta t}{2}\right)^2 = 0.0000390625$ sec.^{2}. By adding this quantity, $\left(\frac{\Delta t}{2}\right)^2$, to the values of (5) for each step, the coefficients of a'' are obtained. By adding this same quantity, $\left(\frac{\Delta t}{2}\right)^2$, to the values of (4) for each step the coefficients of a' are obtained.

The coefficients of v' and d' are dependent on the velocity, and therefore the computations must be repeated for every velocity used in the analysis. These computations are made in the following manner:

(1) Multiply A by $\frac{\pi^2}{L^2} (\dot{x}_P)^2$ and subtract from 1

(2) Multiply (1) by Δt for the velocity used in the analysis

(3) Multiply B by $2 \frac{\pi}{L} \dot{x}_P$.

The values of (1) for each step are the coefficients of d'. The sums of the values of (2) and (3) for each step are the coefficients of v' for each step.

The coefficients of a'', a', v', and d' are tabulated* on the analysis sheets for each step, and are shown underlined on the Sample Analysis Sheet, page 51, which is the first sheet of the analysis for bridge 21 and locomotive J1-D at 5 r.p.s. The remaining part of the preliminary computation is the determination of $\Delta_{D.L.}$, $\Delta_{L.L.}$, and $\Delta_{H.B.}$ for each step.

The dead-load deflection, $\Delta_{D.L.}$, is a constant and in this particular analysis was taken as 0.145 inches.

The computation of $\Delta_{L.L.}$ was made by the ordinary beam theory.

*Columns (2), (6), (7) and (8), page 51.

This theory gives the following expression for the deflection at the center due to a load P a distance $x_P = cL$ from the left end:

$$\text{Center deflection} = \frac{P}{K}(3c - 4c^3)$$

$$\text{for } c < \frac{1}{2}.$$

Using this expression, the center deflection due to the locomotive was computed in the following manner:

(1) The value of c was computed for each wheel on the bridge for each position of the locomotive, determined by the position of the first driver at steps —10, —9, —8, . . . , 0, 1, 2, 3, . . . , 80.

(2) The value of $(3c - 4c^3)$ was tabulated, using table to obtain c^3, for each wheel for steps —10, —9, —8, . . . , 0, 1, 2, 3, . . . , 80.

(3) The values of $(3c - 4c^3)$ at each step were multiplied by the value of $\frac{P}{K}$ for each wheel.

(4) The values of $\frac{P}{K}(3c - 4c^3)$ for all the wheels on the bridge were summed up at each step and $\Sigma \frac{P}{K}(3c - 4c^3) = \Delta_{\text{L.L.}}$ for each step —10, —9, —8, . . . , 0, 1, 2, 3, . . . , 80.

The computation of $\Delta_{\text{H.B.}}$ was made by the ordinary beam theory. The values of $(3c - 4c^3)$ for the driving wheels, 3, 4, and 5, computed for the live-load deflection, were used to determine the center deflection due to a hammer-blow force at a distance x_P from the left end. The hammer-blow force is, however, a variable, and it was necessary to compute the value of the force for each driver for each position of the locomotive. The hammer-blow force is due to the centrifugal effect of the eccentric weights on the drivers. The vertical component of this force varies as the ordinates to a sine curve. As noted before, the total hammer blow of the locomotive is assumed to be divided equally between the three drivers, and the hammer blows of all three are considered to be in phase. The phase angle in this analysis is adjusted to zero degrees so that when the first driver, the reference point, is even with the left end, step 0, the hammer blow will be zero and increasing in magnitude downward. Since the hammer blow has the same value for all drivers and since all drivers are in phase, it will be necessary

to compute the hammer-blow force only for the first driver. This force must be computed for each position of the locomotive, that is, for each step. The length of one step is equal to $\frac{1}{16}$ of the circumference of the driver, and since the hammer-blow force starts at zero, increasing downward, the variation of the force for one revolution will be as follows.

Step	Force
0	H.B. × sin 0 deg. = 0.000 H.B.
1	H.B. × sin 22½ deg. = 0.383 H.B.
2	H.B. × sin 45 deg. = 0.707 H.B.
3	H.B. × sin 67½ deg. = 0.924 H.B.
4	H.B. × sin 90 deg. = 1.000 H.B.
5	H.B. × sin 112½ deg. = 0.924 H.B.
6	H.B. × sin 135 deg. = 0.707 H.B.
7	H.B. × sin 157½ deg. = 0.383 H.B.
8	H.B. × sin 0 deg. = 0.000 H.B.
9	H.B. × sin 202½ deg. = −0.383 H.B.
10	H.B. × sin 225 deg. = −0.707 H.B.
11	H.B. × sin 247½ deg. = −0.924 H.B.
12	H.B. × sin 270 deg. = −1.000 H.B.
13	H.B. × sin 292½ deg. = −0.924 H.B.
14	H.B. × sin 315 deg. = −0.707 H.B.
15	H.B. × sin 347½ deg. = −0.383 H.B.
16	H.B. × sin 0 deg. = −0.000 H.B.

The value of the hammer blow, H.B., varies with the square of the velocity. The value of the total hammer blow for three drivers of J1-D is given as 42 275 pounds at 5 r.p.s.; therefore, the value of H.B. for one driver at any other r.p.s. will be

$$\text{H.B.} = \frac{42\ 275}{3}\left(\frac{\text{r.p.s.}}{5}\right)^2.$$

The computation of the deflection due to the hammer-blow force was made in the following manner.

(1) Beginning with the locomotive in the position designated as step 0 with the first driver at the left end of the bridge, the values of $(3c - 4c^3)$ were summed for each of the three drivers, wheels 3, 4 and 5, at each step.

(2) The values of $\Sigma\ (3c - 4c^3)$ were multiplied at each step by sin (0 deg., 22½ deg., 45 deg., etc.) starting at step 0 with sin 0 deg.

(3) The values of $\Sigma\ (3c - 4c^3)$ sin (0 deg., 22½ deg., 45 deg., etc.) were multiplied at each step by $\frac{\text{H.B.}}{K}$ for the velocity being used.

These values will be the center deflection due to the hammer blow, or

$$\Delta_{\text{H.B.}} = \frac{\text{H.B.}}{K} \Sigma\ (3c - 4c^3) \sin (0 \text{ deg.}, 22\tfrac{1}{2} \text{ deg.}, 45 \text{ deg.}, \text{etc.})$$

The foregoing computations are preliminary to the application of Equation (8) to the step-by-step analysis. The values computed for each step are entered on the computation sheets before beginning the analysis. These values for bridge 21 and locomotive J1-D are underlined on the "Sample Analysis Sheet," page 51. The coefficients of a'' are in column b2,* the coefficients of a' in column c6, the coefficients of v' in column c7, the coefficients of d' in column c8, and the values of $\Delta_{\text{D.L.}} + \Delta_{\text{L.L.}} + \Delta_{\text{H.B.}}$ are in column b10.

For convenience in designating figures of the sample analysis sheet, the first number and letter will designate the line and the last number of the column, for example, in the number —7b6, which is starred on the sample analysis sheet, page 51, —7b indicates line b of step —7 and 6 indicates column (6) of the table. In column 1 are listed the positions of the first driver of the locomotive, starting from a position with the first driver ten steps to the left of the left end of the bridge, and with the first wheel of the locomotive just 8 inches to the left of the end of the bridge. The analysis starts from this position (—10) with the known values of $a' = 0$, $v' = 0$, and $d' = \Delta_{\text{D.L.}} = 0.145$ inches.

The analysis consists of a series of repeated operations associated with each step. There are three parts to each operation as follows:

(1) Compute the right side of Equation (8).

(2) Solve Equation (8) for a'' by dividing the right side by the bracketed quantity on the left side.

(3) Compute the increments to the velocity and deflection in order to obtain new values of v'' and d''.

These new values a'', v'' and d'' become a', v' and d' for the next step, and the operation is repeated.

*Column (2), line b; see following paragraph.

Step —10

1st part:

$$d'\left[1-\frac{\pi^2}{L^2}(\dot{x}_P)^2 A''\right] = \overbrace{-10b5\,[-9c8] = -9b8}^{*}$$

$$v'\left[\Delta t - \Delta t\frac{\pi^2}{L^2}(\dot{x}_P)^2 A'' + 2\frac{\pi}{L}\dot{x}_P B''\right] = -10b4\,[-9c7] = -9b7$$

$$a'\left[\left(\frac{\Delta t}{2}\right)^2 - \left(\frac{C\pi}{32L}\right)^2 A'' + \left(\frac{C\pi}{16L}\right) B''\right] = -10b3\,[-9c6] = -9b6.$$

Right side of equation = (—9b10) – (—9b8) – (—9b7) – (—9b6) = —9b9

2nd part:

Acceleration at end of step,

$$a'' = (-9b9) \text{ divided by } (-9b2) = -9b3.$$

3rd part:

Increment to velocity =

$$[a' + a'']\frac{\Delta t}{2} = [(-10b3) + (-9b3)]\,6.25 = -9a4$$

Increment to deflection =

$$[v' + v'']\frac{\Delta t}{2} = [(-10b4) + (-9b4)]\,0.00625 = -9a5$$

New acceleration $a'' = -9b3$

New velocity $v'' = -9b4$

New deflection $d'' = -9b5$.

Step —9

1st part:

$$d'\left[1-\frac{\pi^2}{L^2}(\dot{x}_P)^2 A''\right] = -9b5\,[-8c8] = -8b8$$

$$v'\left[\Delta t - \Delta t\frac{\pi^2}{L^2}(\dot{x}_P)^2 A'' + 2\frac{\pi}{L}\dot{x}_P B''\right] = -9b4\,[-8c7] = -8b7$$

$$a'\left[\left(\frac{\Delta t}{2}\right)^2 - \left(\frac{C\pi}{32L}\right)^2 A'' + \left(\frac{C\pi}{16L}\right) B''\right] = -9b3\,[-8c6] = -8b6.$$

Right side of equation = (—8b10) – (—8b8) – (—8b7) – (—8b6) = —8b9

*Refers to sample analysis sheet, page 51, where values of the quantities designated by the figures under the bracket are explained. It should be noted that the dash (—) in front of a reference number is part of the step number (—10) and does not indicate that the quantity is to be subtracted.

2nd part:

Acceleration at end of step,

$$a'' = (-8b9) \text{ divided by } (-8b2) = -8b3$$

3rd part:

Increment to velocity =

$$[a' + a''] \frac{\Delta t}{2} = [(-9b3) + (-8b3)]\, 6.25 = -8a4$$

Increment to deflection =

$$[v' + v''] \frac{\Delta t}{2} = [(-9b4) + (-8b4)]\, 0.00625 = -8a5$$

New acceleration = —8b3

New velocity = —8b4

New deflection = —8b5.

These steps are repeated and the arithmetic checked every five steps, using the same operations adapted to the summations of the columns of figures.*

The 9 sheets, pages 52 to 60, show the complete computations for locomotive J1-D crossing bridge 21 at 5 r.p.s. The figures in column b3 are the vertical accelerations of the center of the bridge in inches per second per second, those in column b4 are the vertical velocities of the center in inches per second ($\times$ 1000), and those in column b5 are the total vertical deflections of the center in inches ($\times$ 1000), the values given being for the various steps as indicated. To obtain the dynamic deflection alone at the end of any step, subtract the value of $\Delta_{D.L.} + \Delta_{L.L.}$ (static deflection) from the value of b5. The vertical deflection, velocity, and acceleration of the center of bridge 21 due to locomotive J1-D are shown by the curves of Fig. 5, page 46. The total deflection of the center given in column b10, less the D.L. deflection of 0.145 in., is shown by the upper curve; and the dynamic deflection, hammer blow plus speed effect, is shown by the curve next to the top, Fig. 5b.

*Check each five steps:

1st part: Repeat multiplication;
right side of equation = $\Sigma b10 - \Sigma b8 - \Sigma b7 - \Sigma b6 = \Sigma b9$

2nd part: Repeat division

3rd part: Increment to velocity,
$[(-10b3) + 2\{(-9b3) + (-8b3) + (-7b3)\} + (-6b3)]\, 6.25 = (-6b4) - (-10b4)$

Increment to deflection,
$[(-10b4) + 2\{(-9b4) + (-8b4) + (-7b4)\} + (-6b4)]\, 0.00625 = (-6b5) - (-10b5)$

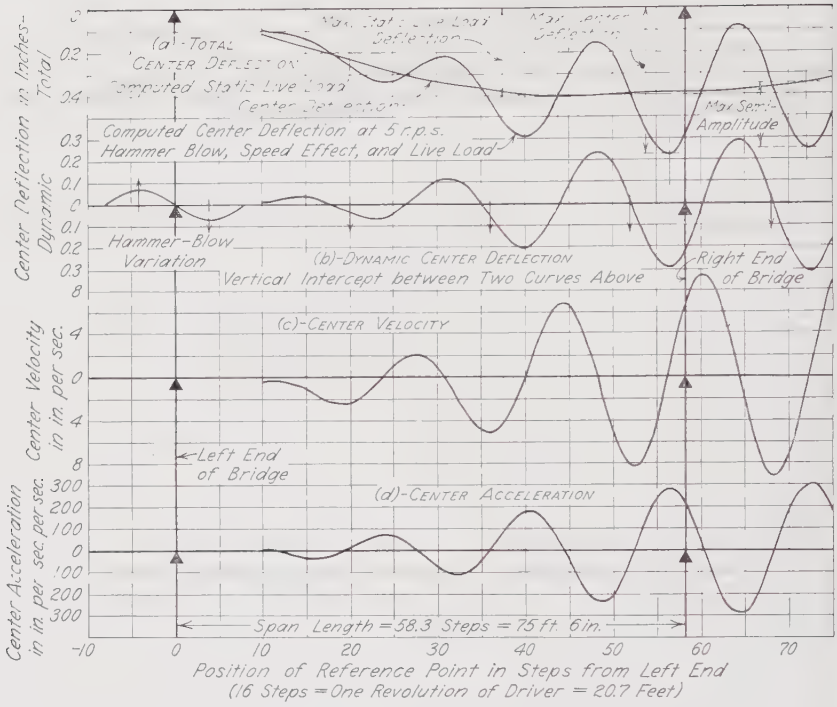


Fig. 5. Deflection, Velocity, and Acceleration of Center of Bridge 21 Due to Locomotive J1-D; Computed Values, Step-by-Step Analysis; 5 r.p.s.; No Damping

In the analysis and the numerical example given, no damping of the structure or locomotive springs was considered, and no differentiation was made between the sprung and the unsprung weight of the locomotive. These, and other desired factors, can be included in the step-by-step analysis, the only limitation being the amount of work necessary to make the computation. The author has made computations including velocity damping of the structure, and analyses to determine the effect of slip of the drivers. The ability to introduce sudden changes of condition into the analysis such as slip of the drivers or rail-joint effect is a particular advantage of the step-by-step analysis. The author has also outlined the analysis to include spring action of the locomotive.

Where the arithmetical work involved becomes too laborious, tests of models similar to those described in Chapter VIII are recommended.

The analysis including bridge damping is as follows: The damping resistance considered will be a force distributed along the bridge proportional to the velocity at each point. The velocity of any point of the bridge at a distance, x_B, from the left end is

$$\frac{dy_B}{dt} = v \sin \frac{\pi x_B}{L}.$$

The damping force is equal to $-c\frac{dy_B}{dt}$ pounds per foot where

$$c = 4\pi n_b m.$$

This damping conforms with the analysis of Inglis for the British 1928 report.

$$\text{Load per foot} = -4\pi n_b m \frac{dy_B}{dt}$$

$$= -4\pi n_b m v \sin \frac{\pi x_B}{L}.$$

The deflection $EIy_B = -4\pi n_b m v \frac{L^4}{\pi^4} \sin \frac{\pi x_B}{L}$, and the center deflection $= -4\pi n_b m v \frac{L^4}{\pi^4} \frac{1}{EI}$.

Introducing

$$\frac{1}{K} = \frac{L^3}{48EI},$$

this becomes

$$-\frac{192 n_b}{\pi^3 g} \frac{wL}{K} v.$$

This increment to the deflection must be added to Equation (7) which then becomes

$$d'' = \Delta_{\text{D.L.}} + \Delta_{\text{L.L.}}'' + \Delta_{\text{H.B.}}'' - \Sigma \frac{P}{K} \frac{\ddot{y}_P''}{g} \sin \frac{\pi x_P''}{L}$$

$$- \frac{1}{2} \frac{wL}{K} \frac{a''}{g} - \frac{192 n_b}{\pi^3 g} \frac{wL}{K} v''. \qquad \text{(7-d)}$$

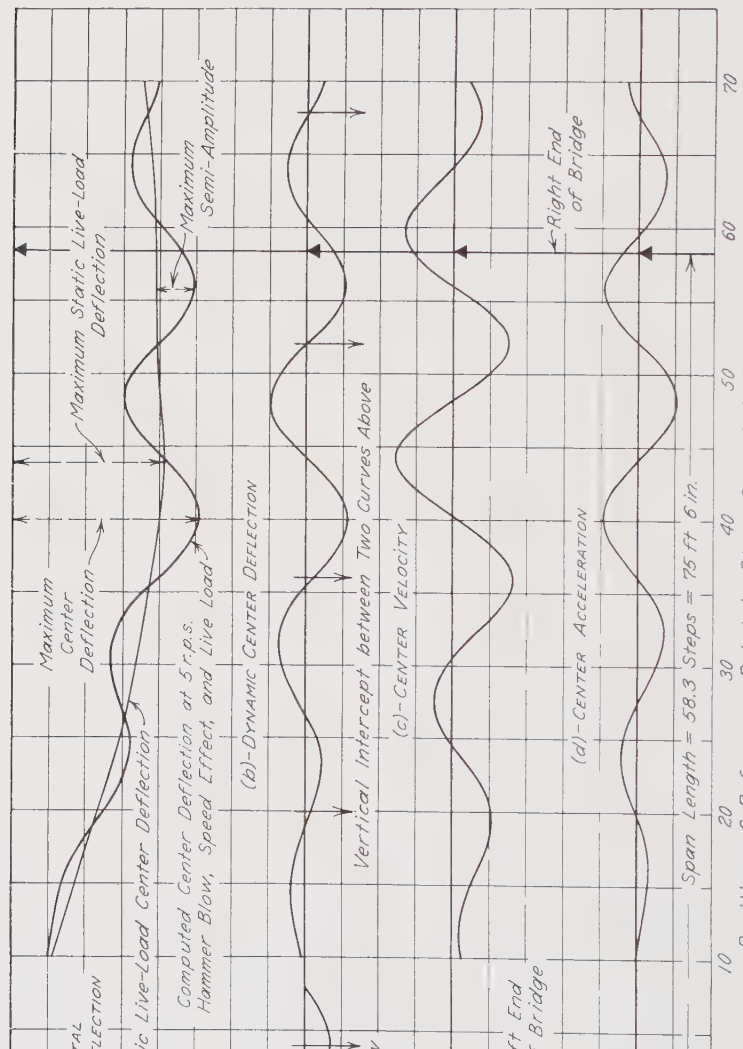


FIG. 6. DEFLECTION, VELOCITY, AND ACCELERATION OF CENTER OF BRIDGE 21 DUE TO LOCOMOTIVE J1-D; COMPUTED VALUES, STEP-BY-STEP ANALYSIS; 5 R.P.S.; BRIDGE DAMPING $n_b = 4.0$

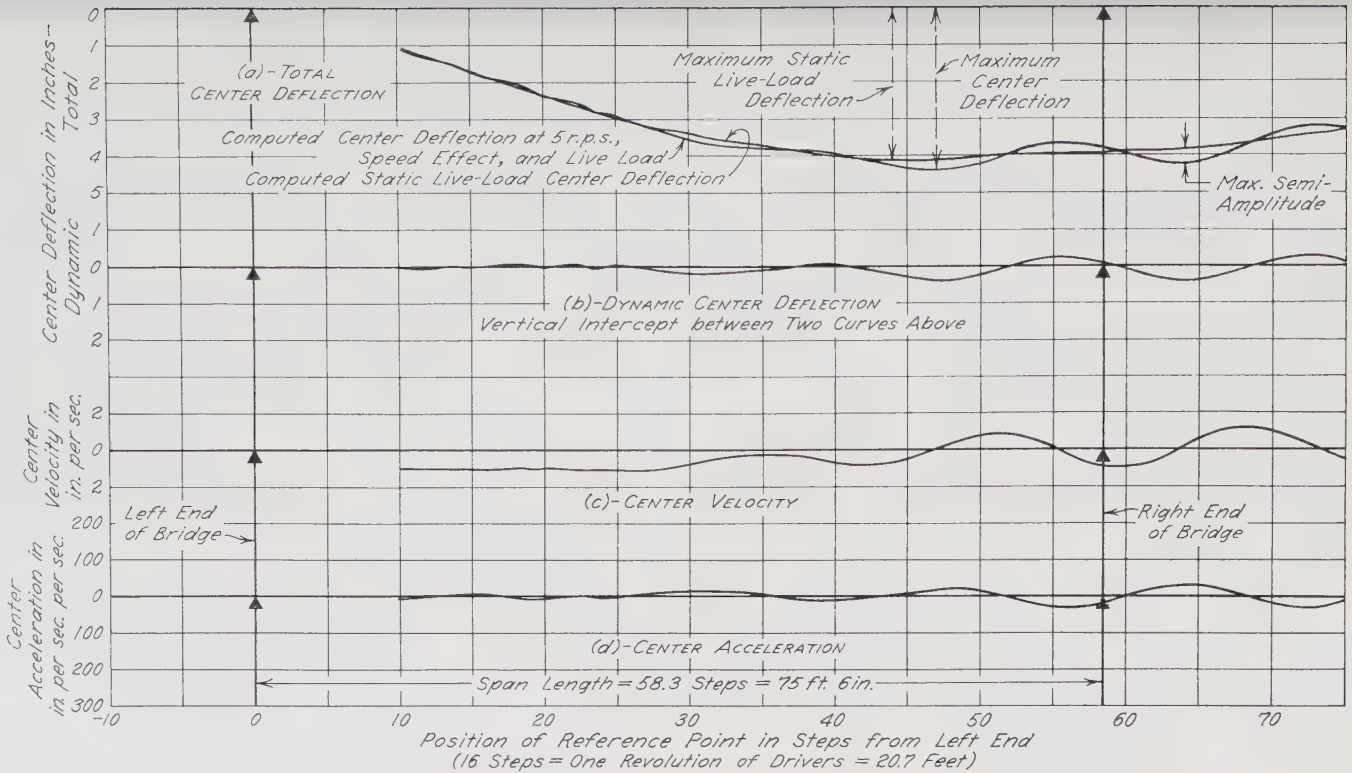


Fig. 7. Deflection, Velocity, and Acceleration of Center of Bridge 21 Due to Locomotive J1-D Without Hammer Blow; Computed Values, Step-by-Step Analysis; 5 r.p.s.; No Damping

It will be necessary to transform v'' into terms of v', a', and a'' before writing Equation (8-D).

$$v'' = v' + a'\frac{\Delta t}{2} + a''\frac{\Delta t}{2}.$$

Therefore

$$-\frac{192n_b}{\pi^3 g}\frac{wL}{K}\left(v' + a'\frac{\Delta t}{2} + a''\frac{\Delta t}{2}\right).$$

The basic equation including velocity damping becomes

$$a''\left[\left(\frac{\Delta t}{2}\right)^2 - \left(\frac{C\pi}{32L}\right)^2 A'' + A'' + \frac{1}{2}\frac{wL}{K}\frac{1}{g} + \left(\frac{C\pi}{16L}\right)B''\right.$$

$$\left. + \frac{192n_b}{\pi^3 g}\frac{wL}{K}\frac{\Delta t}{2}\right] = \Delta_{\text{D.L.}} + \Delta_{\text{L.L.}}'' + \Delta_{\text{H.B.}}'' - d'\left[1 - \frac{\pi^2}{L^2}(\dot{x}_P)^2 A''\right]$$

$$- v'\left[\Delta t - \Delta t\frac{\pi^2}{L^2}(\dot{x}_P)^2 A'' + 2\frac{\pi}{L}\dot{x}_P B'' + \frac{192n_b}{\pi^3 g}\frac{wL}{K}\right]$$

$$- a'\left[\left(\frac{\Delta t}{2}\right)^2 - \left(\frac{C\pi}{32L}\right)^2 A'' + \left(\frac{C\pi}{16L}\right)B'' + \frac{192n_b}{\pi^3 g}\frac{wL}{K}\frac{\Delta t}{2}\right]. \quad \text{(8-D)}$$

The vertical deflection, velocity, and acceleration of the center of bridge 21 due to locomotive J1-D with bridge damping is shown on Fig. 6. The value $n_b = 4.0$ was used in this analysis.

The results of the analysis for this same bridge and locomotive without hammer blow are shown on Fig. 7. This analysis is identical with the analysis illustrated on pages 52 to 60 with the omission of $\Delta_{\text{H.B.}}$ in column 10; see sample analysis sheet, page 51.

SAMPLE ANALYSIS SHEET

Position of First Driver From Left End in Steps	Coefficient of $a'' \times 1000$ (See page 39)	Values of the Acceleration of the Center, a in. per sec.^{2}	Line (a): Values of $(a'-a'')\frac{\Delta t}{2} \times 1000$ Line (b): Values of the Velocity of Center, $v \times 1000$ in. per sec.	Line (a): Values of $(v'-v'')\frac{\Delta t}{2} \times 1000$ Line (b): Values of the Deflection of Center, $d \times 1000$ in.	Line (b): $a' \times$ Coefficient $\times 1000$ Line (c): Coefficient of $a' \times 1000$ (See page 39)	Line (b): $v' \times$ Coefficient $\times 1000$ Line (c): Coefficient of $v' \times 1000$ (See page 39)	Line (b): $d' \times$ Coefficient $\times 1000$ Line (c): Coefficient of $d' \times 1000$ (See page 39)	Values of the Right Side of Equation (8) $\times 1000$	Line (b): Values of $(\Delta_{D.L.} + \Delta_{L.L.} + \Delta_{H.B.}) \times 1000$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
−10 a									
b		0	0	+145.000					
c									
−9 a			+26.6	+0.166					
b	0.248988	+4.26	+26.6	+145.166	+0.0	+0.0	+145.000	+1.060	146.060
c					0.039217	0.012525	0.999998		
−8 a			+96.6	+0.936					
b	0.249874	+11.19	+123.2	+146.102	+0.168	+0.334	+145.164	+2.797	148.046
c					0.039505	0.012571	0.999987		
−7 a			+117.9	+2.277					
b	0.251329	+7.68	+241.1	+148.379	+0.445*	+1.554	+146.097	+1.930	150.026
c					0.039788	0.012616	0.999966		
−6 a			+54.5	+3.354					
b	0.253332	+1.04	+295.6	+151.733	+0.308	+3.052	+148.369	+0.264	151.993
c					0.040061	0.012660	0.999934		

STEP-BY-STEP ANALYSIS

Load: Locomotive J1-D, 13 axles (See Fig. 4, page 37).
Bridge: Bridge 21, 75 foot deck plate girder.
Hammer blow: 42 275 pounds equal to locomotive J1-D.
Speed: 5 r.p.s. or 70 m.p.h.
Dead-load deflection: 0.145 inches.
Static = Column b10 Center velocity = Column b4
Center deflection = Column b5 Center acceleration = Column b3

Column headings same as for Sample Analysis Sheet.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
−10 a									
b		0	0	+145.000					
c									
−9 a			+26.6	+0.166					
b	0.248988	+4.26	+26.6	+145.166	+0.0	+0.0	+145.000	+1.060	146.060
c					0.039217	0.012525	0.999998		
−8 a			+96.6	+0.936					
b	0.249874	+11.19	+123.2	+146.102	+0.168	+0.334	+145.164	+2.797	148.046
c					0.039505	0.012571	0.999987		
−7 a			+117.9	+2.277					
b	0.251329	+7.68	+241.1	+148.379	+0.445	+1.554	+146.097	+1.930	150.026
c					0.039788	0.012616	0.999966		
−6 a			+54.5	+3.354					
b	0.253332	+1.04	+295.6	+151.733	+0.308	+3.052	+148.369	+0.264	151.993
c					0.040061	0.012660	0.999934		
−5 a			−31.8	+3.496					
b	0.255865	−6.13	+263.8	+155.229	+0.042	+3.754	+151.716	−1.569	153.943
c					0.040323	0.012701	0.999891		

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
−4 a			−97.3	+2.689					
b	0.258893	−9.44	+166.5	+157.918	−0.249	+3.361	+155.204	−2.443	155.873
c					0.040570	0.012740	0.999840		
−3 a			−56.1	+1.731					
b	0.262959	+0.47	+110.4	+159.649	−0.388	+2.135	+157.882	+0.123	159.752
c					0.041087	0.012823	0.999774		
−2 a			+62.8	+1.772					
b	0.268022	+9.57	+173.2	+161.421	+0.020	+1.424	+159.599	+2.566	163.609
c					0.041581	0.012901	0.999689		
−1 a			+137.8	+3.026					
b	0.274024	+12.48	+311.0	+164.447	+0.402	+2.247	+161.354	+3.419	167.422
c					0.042046	0.012975	0.999587		
0 a			+127.7	+4.686					
b	0.280891	+7.95	+438.7	+169.133	+0.530	+4.056	+164.359	+2.234	171.179
c					0.042474	0.013043	0.999467		
+1 a			+133.1	+6.316					
b	0.289660	+13.35	+571.8	+175.449	+0.345	+5.788	+169.018	+3.867	179.018
c					0.043419	0.013193	0.999322		
+2 a			+162.9	+8.166					
b	0.300229	+12.72	+734.7	+183.615	+0.592	+7.624	+175.299	+3.818	187.333
c					0.044312	0.013334	0.999143		
+3 a			+118.2	+9.922					
b	0.312472	+6.20	+852.9	+193.537	+0.574	+9.893	+183.419	+1.938	195.824
c					0.045141	0.013466	0.998931		
+4 a			+18.1	+10.774					
b	0.326252	−3.30	+871.0	+204.311	+0.285	+11.587	+193.283	−1.076	204.079
c					0.045899	0.013586	0.998689		
+5 a			−95.5	+10.291					
b	0.341406	−11.98	+775.5	+214.602	−0.154	+11.926	+203.988	−4.090	211.670
c					0.046576	0.013692	0.998421		

STEP-BY-STEP ANALYSIS (CONTINUED)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
+6 a			−136.0	+8.844					
b	0.358282	−9.78	+639.5	+223.446	−0.569	+10.731	+214.200	−3.505	220.857
c					0.047492	0.013837	0.998125		
+7 a			−80.9	+7.488					
b	0.377410	−3.17	+558.6	+230.934	−0.475	+8.954	+222.952	−1.197	230.234
c					0.048538	0.014002	0.997789		
+8 a			−24.4	+6.830					
b	0.398432	−0.73	+534.2	+237.764	−0.157	+7.904	+230.337	−0.290	237.794
c					0.049470	0.014149	0.997417		
+9 a			−16.3	+6.576					
b	0.421108	−1.88	+517.9	+244.340	−0.037	+7.626	+237.053	−0.791	243.851
c					0.050280	0.014276	0.997011		
+10 a			−36.0	+6.249					
b	0.445172	−3.88	+481.9	+250.589	−0.096	+7.448	+243.504	−1.726	249.130
c					0.050957	0.014382	0.996577		
+11 a			−41.1	+5.767					
b	0.470464	−2.70	+440.8	+256.356	−0.200	+6.978	+249.617	−1.268	255.127
c					0.051594	0.014481	0.996120		
+12 a			+22.9	+5.653					
b	0.497770	+6.36	+463.7	+262.009	−0.142	+6.448	+255.236	+3.167	264.709
c					0.052540	0.014629	0.995631		
+13 a			+141.4	+6.680					
b	0.526688	+16.27	+605.1	+268.689	+0.339	+6.840	+260.728	+8.570	276.477
c					0.053327	0.014752	0.995110		
+14 a			+260.2	+9.190					
b	0.556887	+25.36	+865.3	+277.879	+0.878	+8.984	+267.228	+14.124	291.214
c					0.053945	0.014847	0.994561		
+15 a			+356.1	+13.042					
b	0.588014	+31.61	+1221.4	+290.921	+1.379	+12.905	+276.210	+18.587	309.081
c					0.054387	0.014914	0.993992		

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
+16 a			+403.9	+17.792					
b	0.619712	+33.02	+1625.3	+308.713	+1.727	+18.264	+289.004	+20.460	329.455
c					0.054650	0.014953	0.993409		
+17 a			+395.7	+22.789					
b	0.651922	+30.29	+2021.0	+331.502	+1.814	+24.371	+306.496	+19.744	352.425
c					0.054934	0.014995	0.992817		
+18 a			+327.1	+27.307					
b	0.684878	+22.04	+2348.1	+358.809	+1.673	+30.392	+328.920	+15.096	376.081
c					0.055231	0.015038	0.992211		
+19 a			+178.2	+30.465					
b	0.718092	+6.47	+2526.3	+389.274	+1.220	+35.344	+355.794	+4.647	397.005
c					0.055337	0.015052	0.991597		
+20 a			−49.1	+31.272					
b	0.751180	−14.32	+2477.2	+420.546	+0.357	+37.980	+385.763	−10.758	413.342
c					0.055250	0.015034	0.990981		
+21 a			−321.2	+28.958					
b	0.783761	−37.07	+2156.0	+449.504	−0.787	+37.121	+416.497	−29.053	423.778
c					0.054972	0.014985	0.990372		
+22 a			−583.8	+23.301					
b	0.815694	−56.34	+1572.2	+472.805	−2.027	+32.200	+444.907	−45.957	429.123
c					0.054681	0.014935	0.989774		
+23 a			−773.8	+14.816					
b	0.847452	−67.46	+798.4	+487.621	−3.071	+23.434	+467.690	−57.168	430.885
c					0.054515	0.014905	0.989182		
+24 a			−861.5	+4.596					
b	0.878562	−70.38	−63.1	+492.217	−3.654	+11.853	+482.062	−61.830	428.431
c					0.054168	0.014846	0.988599		
+25 a			−835.9	−6.013					
b	0.908661	−63.36	−899.0	+486.204	−3.775	−0.931	+486.326	−57.576	424.044
c					0.053640	0.014758	0.988031		

STEP-BY-STEP ANALYSIS (CONTINUED)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
+26 a			−684.5	−15.516					
b	0.937401	−46.16	−1583.5	+470.688	−3.354	−13.163	+480.119	−43.269	420.333
c					0.052939	0.014642	0.987485		
+27 a			−414.4	−22.384					
b	0.964447	−20.14	−1997.9	+448.304	−2.404	−22.961	+464.554	−19.420	419.769
c					0.052073	0.014500	0.986967		
+28 a			−51.6	−25.296					
b	0.989490	+11.88	−2049.5	+423.008	−1.028	−28.638	+442.244	+11.751	424.329
c					0.051052	0.014334	0.986483		
+29 a			+367.2	−23.324					
b	1.012235	+46.88	−1682.3	+399.684	−0.593	−28.990	+417.103	+47.451	434.971
c					0.049887	0.014145	0.986040		
+30 a			+769.8	−16.218					
b	1.032422	+76.28	−912.5	+383.466	+2.278	−23.443	+393.945	+78.754	451.534
c					0.048592	0.013935	0.985642		
+31 a			+1094.1	−4.568					
b	1.049817	+98.77	+181.6	+378.898	+3.599	−12.508	+377.826	+103.693	472.610
c					0.047182	0.013707	0.985293		
+32 a			+1337.2	+10.628					
b	1.064715	+115.19	+1518.8	+389.526	+4.541	−2.454	+373.212	+122.647	497.946
c					0.045974	0.013512	0.984994		
+33 a			+1384.8	+27.640					
b	1.077449	+106.38	+2903.6	+417.166	+5.164	+20.243	+383.581	+114.620	523.608
c					0.044831	0.013328	0.984737		
+34 a			+1193.4	+43.754					
b	1.087769	+84.57	+4097.0	+460.920	+4.640	+38.130	+410.710	+91.989	545.469
c					0.043616	0.013132	0.984523		
+35 a			+822.0	+56.350					
b	1.095698	+46.95	+4919.0	+517.270	+3.591	+53.036	+453.708	+51.442	561.777
c					0.042457	0.012945	0.984354		

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
+36 a			+283.3	+63.258					
b	1.101991	−1.62	+5202.3	+580.528	+1.953	+62.993	+509.108	−1.781	572.273
c					0.041591	0.012806	0.984221		
+37 a			−365.8	+62.742					
b	1.106486	−56.90	+4836.5	+643.270	−0.066	+65.866	+571.310	−62.961	574.149
c					0.040689	0.012661	0.984121		
+38 a			−1049.5	+53.897					
b	1.109134	−111.02	+3787.0	+697.167	−2.263	+60.514	+633.013	−123.141	568.123
c					0.039763	0.012512	0.984055		
+39 a			−1651.5	+37.016					
b	1.110595	−153.22	+2135.5	+734.183	−4.352	+47.050	+686.024	−170.168	558.554
c					0.039199	0.012424	0.984017		
+40 a			−2064.8	+13.789					
b	1.111125	−177.14	+70.7	+747.972	−5.930	+26.356	+722.435	−196.826	546.035
c					0.038702	0.012342	0.983998		
+41 a			−2216.4	−12.969					
b	1.110653	−177.48	−2145.7	+735.003	−6.767	+0.867	+736.003	−197.118	532.985
c					0.038202	0.012262	0.983998		
+42 a			−2057.3	−39.679					
b	1.109189	−151.69	−4203.0	+695.324	−6.692	−26.141	+723.255	−168.255	522.167
c					0.037706	0.012183	0.984016		
+43 a			−1580.9	+62.418					
b	1.106746	−101.25	−5783.9	+632.906	−5.646	−50.902	+684.235	−112.062	515.625
c					0.037218	0.012111	0.984052		
+44 a			−830.2	−77.488					
b	1.103354	−31.58	−6614.1	+555.418	−3.720	−69.580	+622.847	−34.841	514.706
c					0.036745	0.012030	0.984106		
+45 a			+100.7	−82.047					
b	1.099053	+47.69	−6513.4	+473.371	−1.146	−79.091	+546.630	+52.413	518.806
c					0.036294	0.011958	0.984178		

STEP-BY-STEP ANALYSIS (CONTINUED)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
+46 a			+1081.4	−74.659					
b	1.093890	+125.33	−5432.0	+398.712	+1.710	−77.444	+465.923	+137.100	527.289
c					0.035867	0.011890	0.984265		
+47 a			+1963.4	−55.629					
b	1.087928	+188.82	−3468.6	+343.083	+4.446	−64.250	+392.480	+205.421	538.097
c					0.035471	0.011828	0.984369		
+48 a			+2597.7	−27.122					
b	1.081234	+226.81	−870.9	+315.961	+6.629	−40.829	+337.760	+245.236	548.796
c					0.035110	0.011771	0.984486		
+49 a			+2876.4	+7.091					
b	1.074388	+233.42	+2005.5	+323.052	+7.953	−10.245	+311.099	+250.780	559.587
c					0.035063	0.011764	0.984612		
+50 a			+2738.9	+42.187					
b	1.067339	+204.81	+4744.4	+365.239	+8.156	+23.555	+318.123	+218.604	568.438
c					0.034942	0.011745	0.984741		
+51 a			+2158.9	+72.798					
b	1.060116	+140.62	+6903.3	+438.037	+7.140	+55.666	+359.714	+149.071	571.591
c					0.034861	0.011733	0.984873		
+52 a			+1188.4	+93.719					
b	1.052988	+49.53	+8091.7	+531.756	+4.916	+81.114	+431.470	+52.157	569.657
c					0.034961	0.011750	0.985007		
+53 a									
b	1.046854	−54.30	+8061.9	+632.716	+1.754	+95.668	+523.848	−56.840	564.430
c					0.035414	0.011823	0.985129		
+54 a			−1313.4	+92.565					
b	1.041686	−155.84	+6748.5	+725.281	−1.949	+95.953	+623.373	−162.333	555.044
c					0.035902	0.011902	0.985234		
+55 a			−2451.6	+69.034					
b	1.037243	−236.42	+4296.9	+794.315	−5.647	+80.685	+714.636	−245.223	544.451
c					0.036234	0.011956	0.985323		

STEP-BY-STEP ANALYSIS (CONTINUED)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
+56 a			−3225.6	+33.551					
b	1.033996	−279.68	+1071.3	+827.866	−8.722	+51.825	+782.714	−289.188	536.629
c					0.036890	0.012061	0.985395		
+57 a			−3494.8	−8.451					
b	1.032229	−279.49	−2423.5	+819.415	−10.519	+13.045	+815.813	−288.497	529.842
c					0.037610	0.012177	0.985441		
+58 a			−3214.6	−50.385					
b	1.031919	−234.84	−5638.1	+769.030	−10.716	−29.794	+807.502	−242.331	524.661
c					0.038340	0.012294	0.985461		
+59 a			−2411.4	−85.548					
b	1.032483	−150.98	−8039.5	+683.482	−9.091	−69.647	+757.846	−155.880	523.228
c					0.038712	0.012353	0.985457		
+60 a			−1209.3	−108.177					
b	1 033260	−42.51	−9258.8	+575.305	−5.870	−99.653	+673.535	−43.925	524.087
c					0.038882	0.012380	0.985446		
+61 a			+188.4	−114.558					
b	1.034371	+72.66	−9070.4	+460.747	−1.660	−114.874	+566.922	+75.153	525.541
c					0.039048	0.012407	0.985428		
+62 a			+1556.0	−103.655					
b	1.035804	+176.30	−7514.4	+357.092	+2.849	−112.763	+454.022	+182.615	526.723
c					0.039208	0.012432	0.985405		
+63 a			+2677.2	−77.198					
b	1.037538	+252.06	−4837.2	+279.894	+6.939	−93.599	+351.870	+261.518	526.728
c					0.039358	0.012456	0.985375		
+64 a			+3380.7	−39.336					
b	1.039398	+288.85	−1456.5	+240.558	+9.923	−60.262	+275.791	−300.225	525.677
c					0.039369	0.012458	0.985341		
+65 a			+3579.6	+4.166					
b	1.040308	+283.88	+2123.1	+244.724	+11.249	−18.045	+237.026	+295.324	525.554
c					0.038943	0.012389	0.984316		

STEP-BY-STEP ANALYSIS (CONCLUDED)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
+66 a			+3248.2	+46.840					
b	1.040356	+235.83	+5371.3	+291.564	+10.933	+26.157	+241.128	+245.346	523.564
c					0.038512	0.012320	0.985307		
+67 a			+2421.9	+82.278					
b	1.039544	+151.67	+7793.2	+373.842	+8.971	+65.804	+287.282	+157.670	519.727
c					0.038082	0.012251	0.985315		
+68 a			+1220.6	+105.044					
b	1.037878	+43.62	+9013.8	+478.886	+5.711	+94.952	+368.361	+45.277	514.301
c					0.037657	0.012184	0.985338		
+69 a			−183.8	+111.524					
b	1.035381	−73.03	+8830.0	+590.410	+1.624	+109.761	+471.883	−75.612	507.656
c					0.037242	0.012177	0.985376		
+70 a			−1555.8	+100.651					
b	1.031501	−175.90	+7274.2	+691.061	−2.676	+106.154	+581.812	−181.440	503.850
c					0.036641	0.012022	0.985437		
+71 a			−2690.8	+74.110					
b	1.025161	−254.62	+4583.4	+765.171	−6.237	+86.068	+681.063	−261.029	499.865
c					0.035455	0.011832	0.985533		
+72 a			−3459.8	+35.669					
b	1.017278	−298.94	+1123.6	+800.840	−8.800	+53.580	+754.200	−304.107	494.873
c					0.034561	0.011690	0.985662		
+73 a			−3750.7	−9.397					
b	1.007687	−301.17	−2627.1	+791.443	−10.078	+12.984	+789.487	−303.490	488.903
c					0.033714	0.011556	0.985824		
+74 a			−3503.4	−54.735					
b	0.996501	−259.38	−6130.5	+736.708	−9.916	−30.028	+780.376	−258.472	481.960
c					0.032924	0.011430	0.986017		
+75 a			−2729.2	−93.689					
b	0.983776	−177.30	−8859.7	+643.019	−8.336	−69.311	+726.570	−174.429	474.494
c					0.032137	0.011306	0.986239		

VII. Application of Step-by-Step Analysis

24. *Introduction.*—The step-by-step analysis was described in Chapter VI, and illustrated by the analysis of locomotive J1-D crossing bridge 21. Field tests have also been made on this locomotive and bridge.* The present chapter contains further studies to determine the possibilities and limitations of the step-by-step method in which the results of the analysis given in Chapter VI will be used as the basis of comparison.

One important factor in considering an arithmetical solution is the amount of work involved in making the computations. The amount of work can be reduced by arranging the computations as suggested in the previous chapter. The choice of Δt as a function of the r.p.s. also reduces the work by making a large part of the preliminary computations independent of the velocity of the locomotive. Thus analyses can be made for several velocities without repeating these preliminary computations.

The length of the time interval, Δt, requires careful consideration. It must be kept small enough to reduce the error resulting from the substitution of small finite for differential increments, Fig. 2, p. 30, but the smaller the time interval the more work there will be in making the computations. The interval used was chosen after making parallel computations using values of Δt of 1/(8 r.p.s.), 1/(16 r.p.s.), and 1/(32 r.p.s.). Two center-deflection diagrams are given in Fig. 8, page 62; for one, $\Delta t = 1/(16$ r.p.s.) and, for the other, $\Delta t = 1/(32$ r.p.s.). The computed deflections based upon a time interval of 1/(8 r.p.s.) differed so much from those based on a time interval of 1/(16 r.p.s.) that a value of 1/(8 r.p.s.) was considered to be too large, but the small difference resulting from reducing the time interval from 1/(16 r.p.s.) to 1/(32 r.p.s.) would apparently justify the use of the former. Therefore a time interval, Δt, of 1/(16 r.p.s.) has been used.

Analyses were also made using the first terms only of each of Equations (3), page 29. This was equivalent to assuming that the motion of the wheels of the locomotive was identical with the motion of the points of the beam immediately beneath them. This is apparent if x_B is substituted for x_P; then the first terms of Equations (3) become equal to Equations (2). It was found that for speeds below 5 r.p.s. (70 m.p.h.) the difference was small. Above this speed the values of the terms containing $\dot{x}_P$ were too large to be

*"Impact on Steel Railway Bridges of Simple Span," J. B. Hunley, Proceedings, A.R.E.A., Vol. 37 (1936).

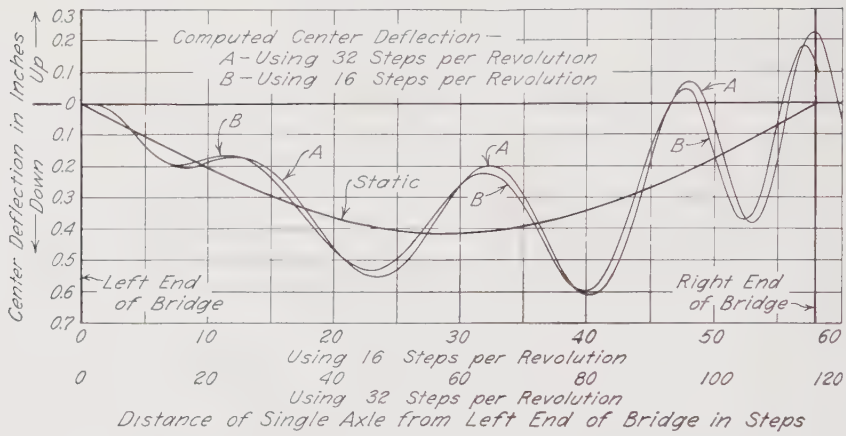


Fig. 8. Comparison of Computed Deflections of Center of Bridge 21 Due to Single-Axle Load, Using 16 and 32 Steps per Revolution in Computations

neglected. A comparison of the different terms of Equations (3) is shown on Fig. 9, page 63. It was decided to include the terms containing $\dot{x}_P$ in the computations.

25. *Analyses of Equivalent Locomotives.*—The analysis of the locomotive and bridge given in Chapter VI took into account all thirteen axles, and the results have been used as a basis in judging whether or not it is feasible to use an "equivalent locomotive" instead of the actual locomotive in analyses to determine dynamic effects upon a bridge. An "equivalent locomotive" as the term is used here, is a locomotive with a smaller number of axles than the actual locomotive, that will produce dynamic stresses in the bridge sufficiently equal, for purposes of design, to the stresses produced by the actual locomotive. Locomotive J1-D was used as the prototype for two equivalent locomotives; one equivalent locomotive had five axles, and the other had one axle. The deflections of bridge 21,* due to these two equivalent locomotives have been computed, and the dynamic deflections for these have been compared with the dynamic deflection of the same bridge due to locomotive J1-D. Data on bridge 21 and locomotive J1-D can be found on page 36 and on Fig. 4, page 37. The results of the analysis are shown on Fig. 5.

*The bridge used in the analysis given in Chapter VI.

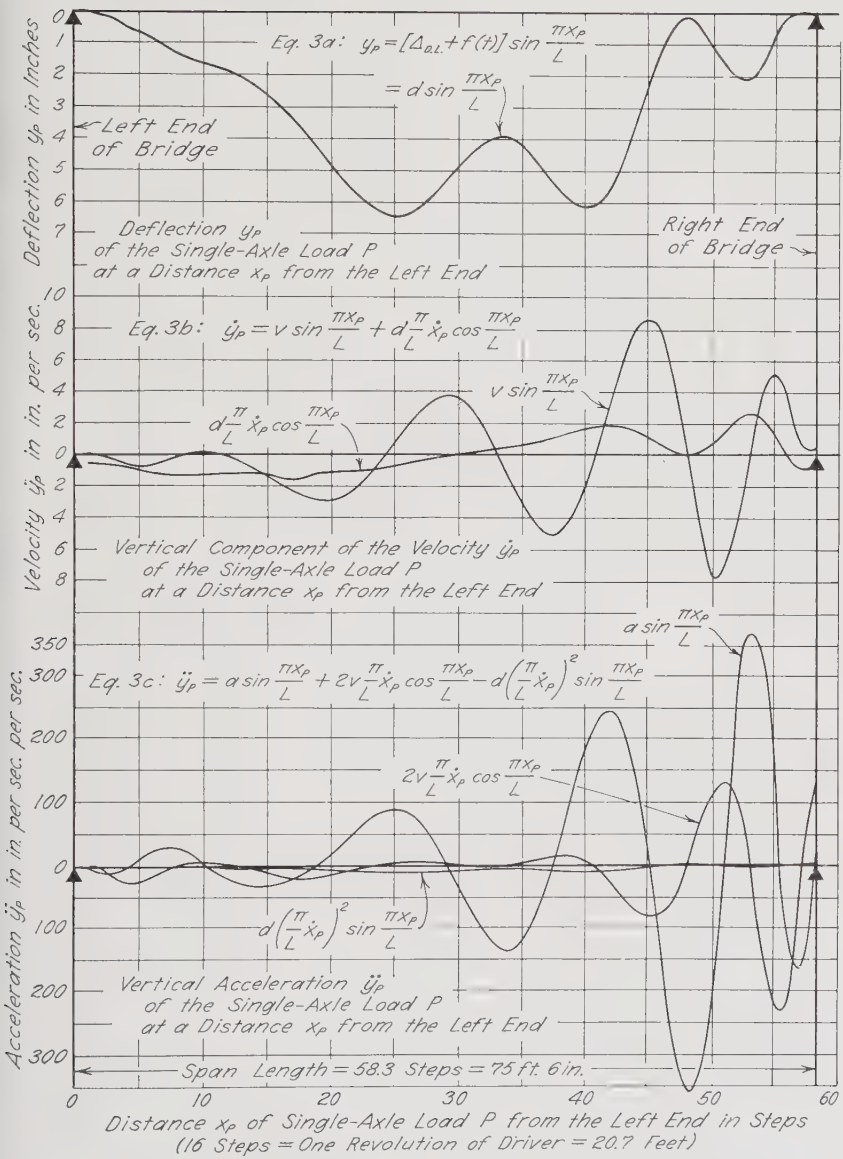


FIG. 9. COMPARISON OF VALUES OF TERMS IN EQUATIONS (2) AND (3). DEFLECTION, VELOCITY, AND ACCELERATION OF BRIDGE 21 AT A DISTANCE x_p FROM THE LEFT END; SINGLE-AXLE LOCOMOTIVE EQUIVALENT TO LOCOMOTIVE J1-D; COMPUTED VALUES, STEP-BY-STEP ANALYSIS; 5 R.P.S.; NO DAMPING

Five-Axle Equivalent Locomotive

The distance between the axles and the load carried on each axle are given in Fig. 4a, page 37, for locomotive J1-D and in Fig. 4b for the five-axle equivalent. The first axle load of the five-axle equivalent is the resultant of the loads on the front truck of locomotive J1-D; the second axle load is the resultant of the loads on the three drivers; the third axle load is the resultant of the loads on the rear truck of the locomotive, and the loads on the fourth and fifth axles are the resultants of the loads on the two trucks of the tender. The circumference of the single driver of the five-axle equivalent locomotive was the same as for J1-D; and the hammer blow for the single driver of the five-axle equivalent was equal to the total hammer blow for the three drivers of J1-D, or 42 275 pounds at 5 r.p.s.

The reference point that determines the position of the five-axle locomotive on the bridge is a point 7 feet (5.4 steps) in front of the single driver, axle 2. The magnitude of the hammer-blow force for any position of the five-axle locomotive is determined by the angular position of the counterweight on the single driver when the reference point is at the left end of the bridge. In this analysis the phase angle was zero; this made the hammer-blow force zero when the reference point was at the left end of the bridge, and increasing in magnitude downward as the five-axle locomotive moved forward.* This method of expressing the position and phase angle is convenient in making the transition from prototype to a model as will be done in Chapter VIII.

The procedure for analyzing the five-axle equivalent locomotive was the same as that used for the analysis of J1-D, described in Chapter VI.

The total deflection, velocity, and acceleration of the center of the bridge during the passage of the five-axle equivalent locomotive, as determined by the step-by-step analysis, are given by the curves of Fig. 10. The dynamic deflection due to combined hammer blow and speed effect is represented by the vertical intercept between the two curves at the top of the figure, one representing total center deflection and the other representing the static deflection due to live load. This dynamic deflection of the center as well as the velocity and acceleration are also shown in the figure. Curves similar to those in Fig. 10 are given in Fig. 5 for locomotive J1-D.

The deflection due to speed effect for a five-axle locomotive, similar to the five-axle equivalent of J1-D except that the driver had no

*See page 38 for details of parallel description for locomotive J1-D.

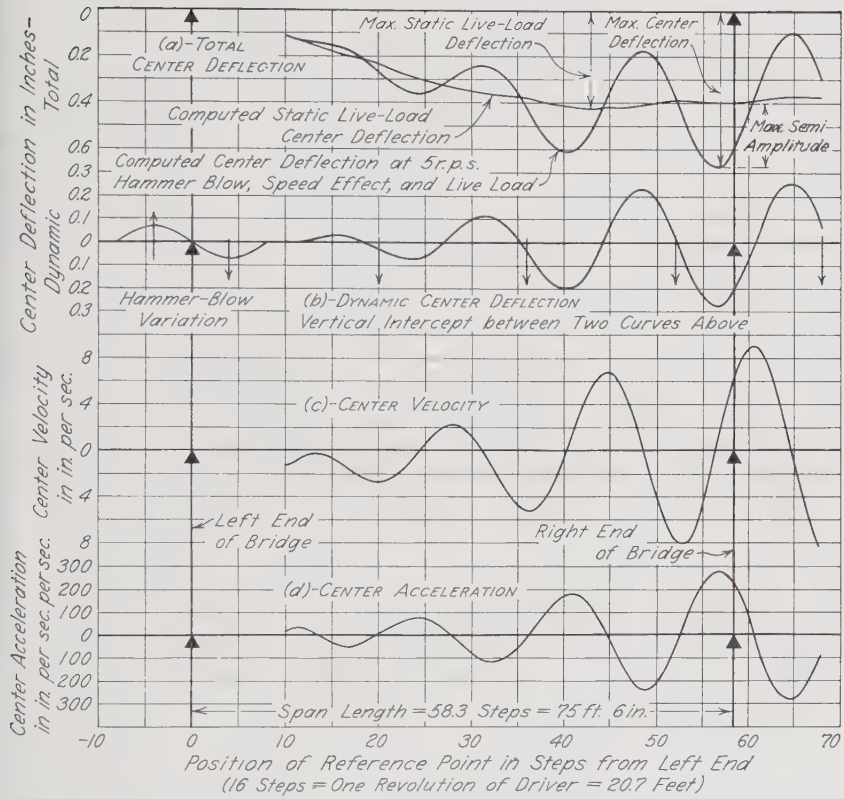


Fig. 10. Deflection, Velocity, and Acceleration of Center of Bridge 21 Due to Five-Axle Locomotive Equivalent to Locomotive J1-D; Computed Values, Step-by-Step Analysis; 5 r.p.s.; No Damping

counterweight (five smoothly rolling wheels), computed with the step-by-step analysis, is shown by the curves of Fig. 11.

Single-Axle Equivalent Locomotive

If a locomotive is assumed to be represented by a single-axle load with hammer blow, the assumption as to bridge and load in this analysis will coincide with those made in the analysis by Inglis.*

The single concentrated load was made equivalent to locomotive J1-D, and the bridge data were the same as for bridge 21.

The single-axle load equivalent to the locomotive was taken, as suggested by Hunley,† to be a single load which, when placed at the

*See footnote, page 11.
†See footnote, page 16.

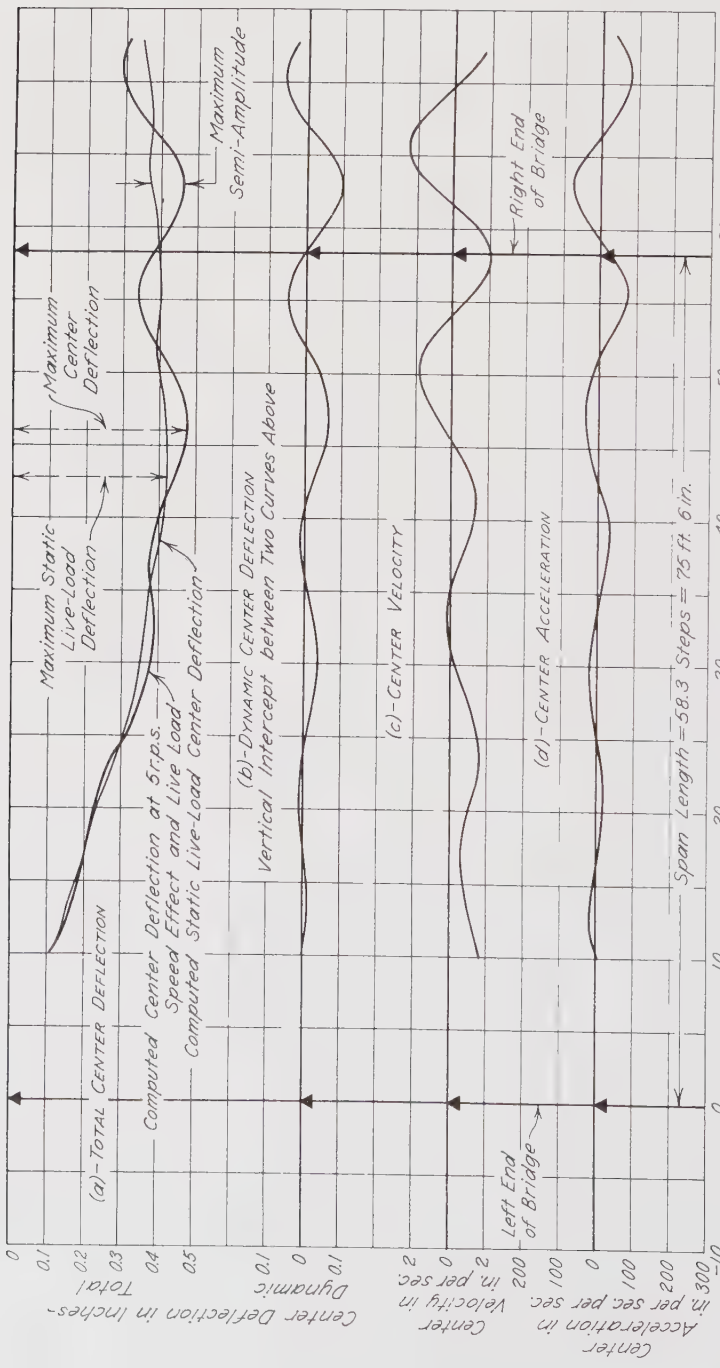


FIG. 11. DEFLECTION, VELOCITY, AND ACCELERATION OF CENTER OF BRIDGE 21 DUE TO FIVE-AXLE LOCOMOTIVE EQUIVALENT TO LOCOMOTIVE J1-D WITHOUT HAMMER BLOW; COMPUTED VALUES, STEP-BY-STEP ANALYSIS; 5 R.P.S.; NO DAMPING

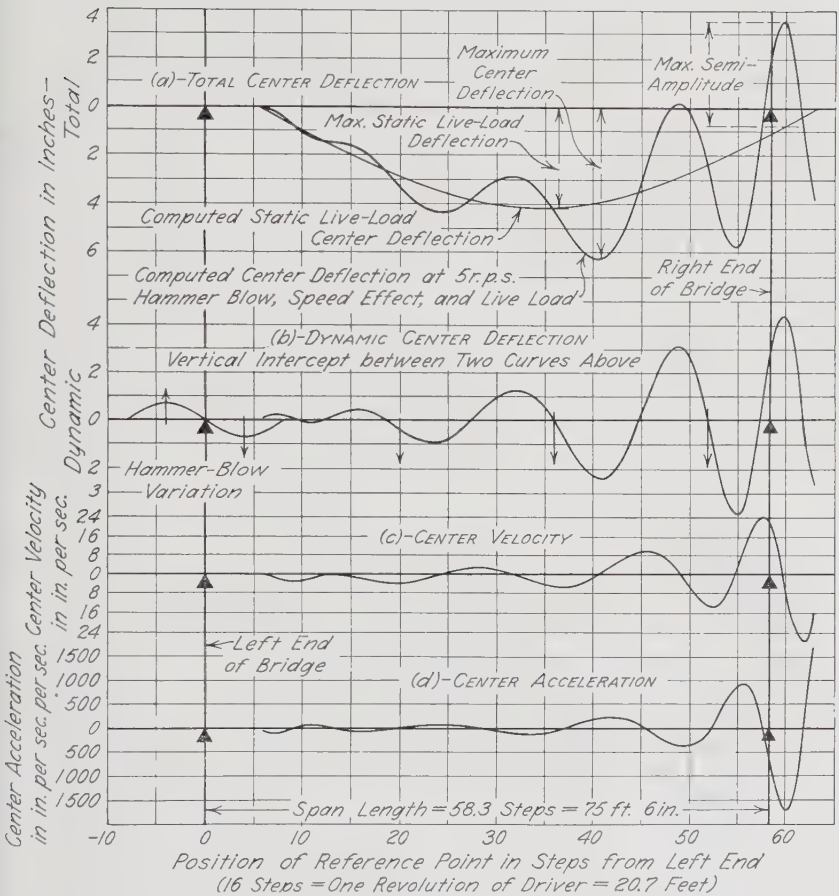


Fig. 12. Deflection, Velocity, and Acceleration of Center of Bridge 21 Due to Single-Axle Locomotive Equivalent to Locomotive J1-D; Computed Values, Step-by-Step Analysis; 5 r.p.s.; No Damping

center of the span, would cause the same maximum static center deflection as the locomotive. This maximum deflection was assumed to occur when the locomotive was in the position for maximum moment. The position of locomotive J1-D for maximum moment was determined, and the static deflection of the center of bridge 21 due to the locomotive in that position was found to be 0.419 inches. Since the modulus of the bridge was 840 000 pounds per inch, the weight of the single-axle equivalent was 840 000 × 0.419, or 350 000 pounds.

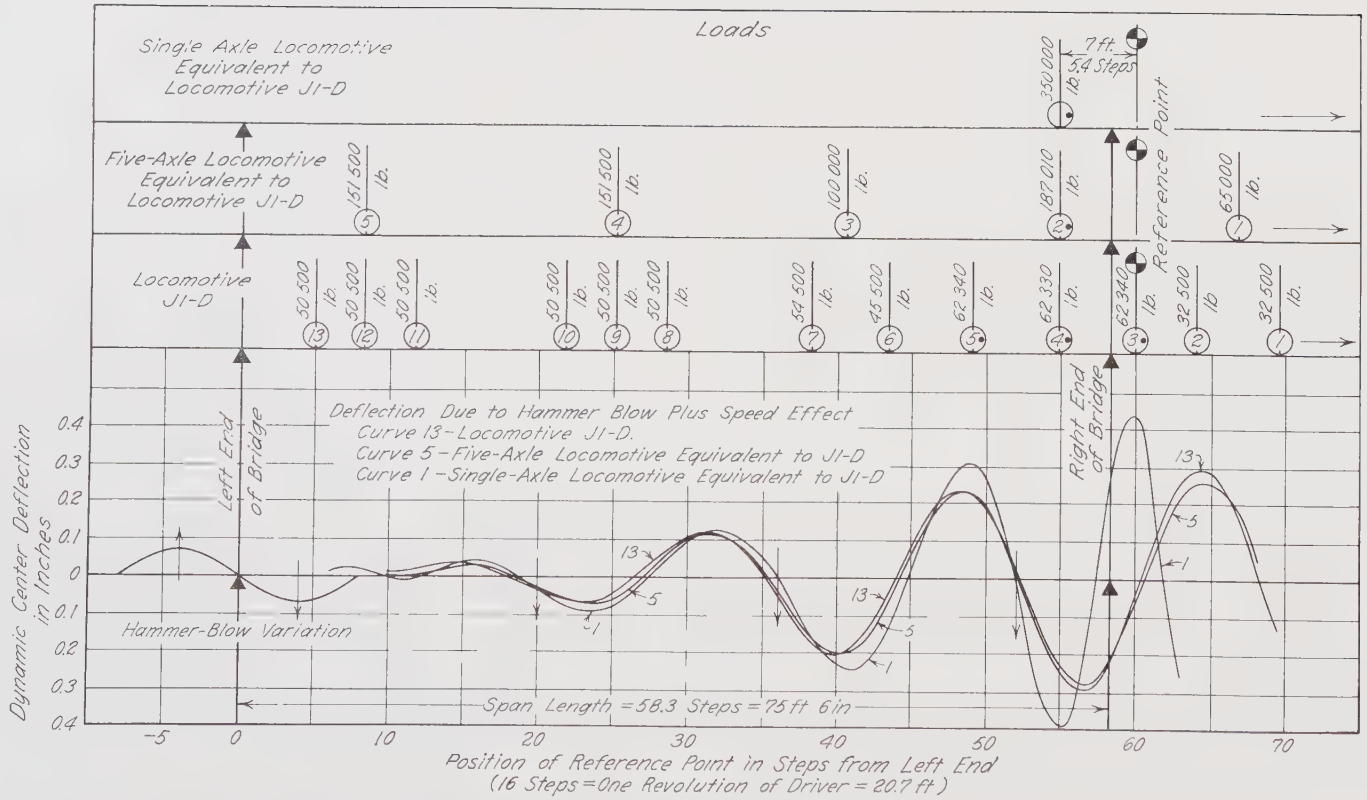


FIG. 13. DYNAMIC CENTER DEFLECTION OF BRIDGE 21 DUE TO LOCOMOTIVE J1-D, TO THE FIVE-AXLE LOCOMOTIVE EQUIVALENT TO J1-D, AND TO THE SINGLE-AXLE LOCOMOTIVE EQUIPMENT TO J1-D; STEP-BY-STEP ANALYSIS; 5 R.P.S.; NO DAMPING

The circumference of the single wheel of the single-axle equivalent locomotive was the same as that of the drivers of J1-D. The hammer blow of the single-axle equivalent was taken as equal to the total hammer blow of locomotive J1-D, or 42 275 pounds at 5 r.p.s. The phase angle position of the counterweight on the single driver was zero so that the hammer blow would be zero and increasing in magnitude downward when the reference point, which is 7 feet (5.4 steps) ahead of the axle, was at the left end of the bridge. This phase angle and reference point are consistent with the similar data used in the analyses of J1-D and the five-axle equivalent locomotive. The procedure in making the analysis was the same as that used for the analysis of J1-D, described in detail in Chapter VI.

The total deflection, velocity, and acceleration of the center of bridge 21 during the passage of the single-axle equivalent locomotive, as determined by the step-by-step analysis, are given by the curves of Fig. 12. Similar curves for locomotive J1-D and the five-axle equivalent are given in Figs. 5 and 10, respectively.

The dynamic center deflections of bridge 21 due to the passage of locomotive J1-D, of the five-axle equivalent of J1-D, and of the single-axle equivalent of J1-D are shown by the superimposed curves of Fig. 13. The correct corresponding center deflections will be obtained when the second driver of locomotive J1-D, (wheel 4), the driver of the five-axle equivalent locomotive, (wheel 2), and the single axle of the single-axle equivalent locomotive are all at the same distance from the left end of the bridge. The reference point of locomotive J1-D, which locates the distance of J1-D from the left end of the bridge, is the first driver, (wheel 3). The reference point for the five-axle equivalent locomotive must, therefore, be a point 7 feet (5.4 steps) in front of wheel 2. The reference point of the single-axle equivalent locomotive must be a point of 7 feet (5.4 steps) in front of the single axle.

The phase angle of the hammer blow must be the same for all these locomotives. This was accomplished by making the phase angle zero, and thus the hammer blow was zero and increasing in magnitude downward when the reference point of each locomotive was at the left end of the bridge. It is apparent from these curves that the deflection due to the five-axle equivalent locomotive approximates very closely the deflection due to J1-D, but that the maximum deflection due to the single-axle equivalent locomotive is considerably greater than the deflection due to J1-D.

VIII. Determination of Impact by Laboratory Tests of Models

26. *Description of Apparatus.*—The model bridge on which tests were made consisted of a steel bar simply supported at the ends. The model locomotive was a series of wheels in a single plane, counterweighted below the track for equilibrium. The hammer blow was obtained by means of an eccentric weight on one of the wheels. The rules of dynamic similitude were observed between the model and prototype.

The apparatus, which is shown in Figs. 14 and 15, was so designed that it was possible to change and study the following factors independently:

Bridge Factors

(a) Span length
(b) Weight per foot
(c) Stiffness
(d) Damping force proportional to velocity.

Locomotive Factors

(a) Velocity
(b) Spacing and weight of wheels
(c) Hammer blow.

Although it was possible to vary all of the factors listed, only a few of these factors were studied in the tests reported in this bulletin. The following tests were made:

One model bridge was tested under various damping forces, all proportional to the vertical velocity of the center of the bridge.

Two model locomotives, one a single-axle load and the other a series of five-axles, were tested at various velocities and with various hammer blows.

The testing procedure consisted in rolling the wheel or combination of wheels over the steel bar at various velocities and making an autographic record of the center deflection of the bar.

The information obtained was as follows:

(1) Magnitude of static load and hammer blow

(2) Length, weight, and stiffness of bridge

(3) Velocity of load expressed in terms of revolutions per second of the wheel representing the drivers of the locomotive

(4) Center deflection; a continuous record while load was crossing bridge

(5) Position of load on bridge relative to the deflection diagram

(6) Location of points of maximum hammer blow relative to deflection diagram

FIG. 14. APPARATUS USED FOR LABORATORY TEST OF MODELS

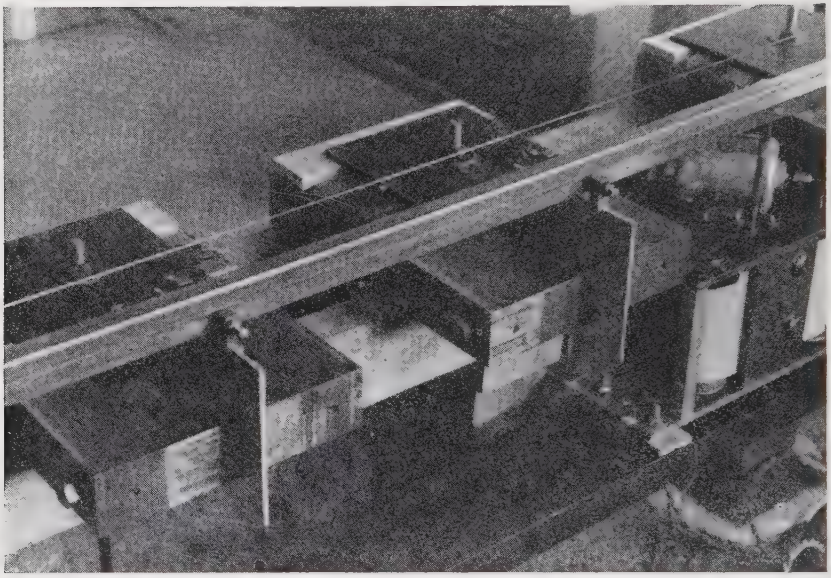


FIG. 15. APPARATUS FOR PRODUCING DAMPING OF BRIDGE PROPORTIONAL TO VERTICAL VELOCITY IN MODEL TESTS

(7) Experimentally-determined damping force for bridge obtained from a record showing the rate at which free vibrations diminish.

Items (1) and (2) were computed; items (3), (4), (5), (6), and (7) were obtained from a deflectometer record. The record was produced by means of a metal scribe bearing on a strip of specially-prepared waxed paper.

The model bridge was proportioned to represent bridge 21 of the A.R.E.A. 1936 report. The locomotive models were proportioned to represent locomotive J1-D of the same report.

The characteristics of both the bridge and the locomotive prototypes are given in Sections 23 and 25. The model bridge had for its prototype a 75.5-foot open-deck, single-track, plate-girder bridge. The prototype for the single-axle model was the single-axle equivalent of locomotive J1-D; and for the five-axle model the prototype consisted of the five-axle equivalent to J1-D.*

*One slight difference existed between the single- and five-axle equivalents of locomotive J1-D described in Section 25, and the prototypes of the models. The hammer blow of the equivalent locomotives used in the step-by-step analysis was 42 275 pounds at 5 r.p.s., which is the total hammer blow of locomotive J1-D. This total hammer blow of locomotive J1-D is divided into three forces on the three drivers 7 feet apart. The single force which will cause the same maximum static center deflection as these three separate forces is 41 300 pounds at 5 r.p.s. The hammer blow of the single- and five-axle equivalents of J1-D was concentrated on one driving axle, so the hammer blow of the prototype was taken as 41 300 pounds instead of 42 275 pounds in the design of the models. No other difference existed.

The following suggestions relative to the design of a model of a locomotive and bridge are offered:

There are three important scales to consider in designing this model apparatus; these are the length of the model bridge, the weight of the model locomotive, and the magnitude of the center deflection of the model bridge.

The bridge length ratio of the prototype to the model determines the velocity ratio. For prototype velocities as high as 100 m.p.h. (150 f.p.s.) the model velocity must be low enough to be controlled. In these tests the model velocity was kept below 10 feet per second. The locomotive weight ratio of the prototype to model involves a matter of judgment as to the strength and rigidity of the whole apparatus. The weight of the model must also be low enough to control at a velocity of 10 feet per second (or whatever maximum velocity is used). The model bridge deflection must be large enough to be measured with a certain degree of accuracy. For example, speed effect is about 10 per cent of the total deflection. In order that the speed effect be measured with an accuracy of less than 10 per cent error, the total deflection must be measured with an accuracy of less than 1 per cent error.

The conditions of similitude can be obtained from Equation (8), page 34 of the step-by-step analysis. The condition for similitude between a model and the prototype is that certain quantities in this equation shall have the same value for the model as for the prototype. These quantities are Δt, $\frac{C}{L}$, $\frac{P}{K}$, $\frac{wL}{K}$, $\frac{x_P}{L}$, $\frac{\dot{x}_P}{L}$, $\Delta_{D.L.}$, $\Delta_{L.L.}$, and $\Delta_{H.B.}$ The model was designed to meet this requirement.

The data for the single-axle prototype and model are as follows:

Single-Axle Model

Locomotive	Prototype	Model
A single-axle load	350 000 pounds	46.82 pounds
Hammer blow at 5 r.p.s.	41 300 pounds*	5.522 pounds
Circumference of driver	20.7 feet	19.53 inches
Bridge	**Prototype**	**Model**
Length	75.5 feet	71.3 inches
Weight	136 000 pounds	18.18 pounds
Modulus	840 000 pounds per inch	112.3 pounds per inch

*See footnote, page 72.

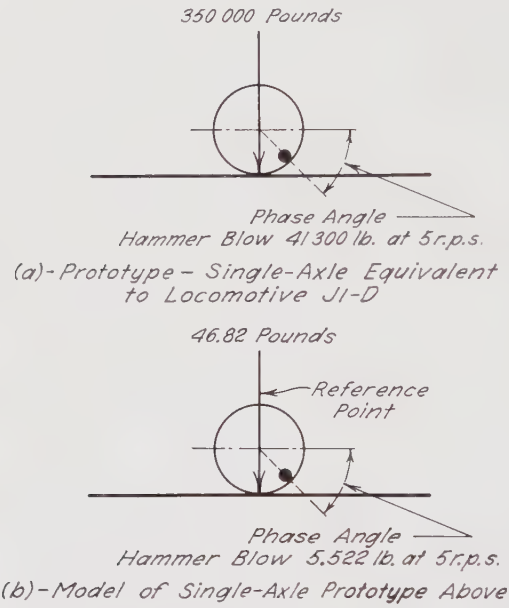


FIG. 16. PROTOTYPE AND MODEL OF SINGLE-AXLE LOCOMOTIVE EQUIVALENT TO LOCOMOTIVE J1-D

The model apparatus shown in Figs. 14 and 15 consists of six essential parts, as follows:

(1) Locomotive
(2) Bridge
(3) Reference frame
(4) Deflectometer
(5) Track
(6) Magnets for producing damping.

The single-axle model shown in Fig. 16 consisted of a wheel with an attached hanger-arm on which weights were so hung as to bring the center of gravity of the moving mass below the top of the rail, thus producing stable equilibrium. This is shown in the lower right-hand corner of Fig. 14. The combined weight of the wheel, hanger-arm, and weights constituted the weight of the locomotive model. The counterweight, which consisted of two discs, one on each side of the wheel, as shown in the figure, was located and proportioned in such a way as to comply with the conditions for similitude. The model wheel circumference and length of bridge were proportioned to give a reasonable velocity of the model along the track; also the conditions of similitude

were satisfied so that a given r.p.s. of the prototype was equivalent to the same r.p.s. of the model. This wheel, which represented the locomotive, was connected to a similar rear wheel by means of two bamboo poles, as shown in the figure. The rear wheel, which was counterweighted in the same way as the front wheel except that the two counterweights were 180 degrees out of phase, served as a guide to hold the front wheel in line. It also served to balance horizontally the hammer-blow force on the front wheel. The distance between the two wheels was such that the rear wheel was on the horizontal approach while the front wheel, the model of the locomotive, was crossing the model bridge.

The model bridge was a rectangular steel bar $1\frac{7}{16}$ inches x $\frac{5}{8}$ inches, in the flat position. One end was supported by a pin in fixed roller bearings, and the other by a pin in roller bearings which rested on a polished horizontal steel plate. This arrangement was used to minimize the damping effect of the bearings. The rail consisted of a 12 gauge steel wire on the top of the steel bar.

A reference frame consisting of a steel channel supported from the ends of the bridge model, carried the deflectometer, as shown in the lower part of Fig. 14. A metal stylus attached to the mid-point of the bridge bore upon a strip of waxed paper $4\frac{1}{4}$ inches wide. The rolls on which this strip of paper was wound were rotated by an electric motor, so that the stylus traced a line on the paper which indicated the deflection of the center of the bridge.

There were two electrical contacts on the frame of the model locomotive. One circuit was closed by a pin on the wheel situated diametrically opposite the counterweight. This circuit was closed during each revolution of the wheel when the counterweight was at the same height and in front of the axle, thus indicating the position of the wheel on the bridge when the hammer blow was zero and increasing in magnitude downward.* The other circuit was closed by two fixed contacts located exactly in line with the ends of the bridge, thus indicating the position of the locomotive on the deflection diagram relative to the ends of the bridge.

A sample record is shown in Fig. 17. The complete record is approximately $4\frac{1}{4}$ inches by 60 inches. It consists of four lines. The upper line, with its notches, indicates the position of the model locomotive relative to the ends of the bridge and the instant when the phase angle of the counterweight was zero. The second line from the top is a continuous autographic record of the vertical deflection of the

*Phase angle zero, see page 38.

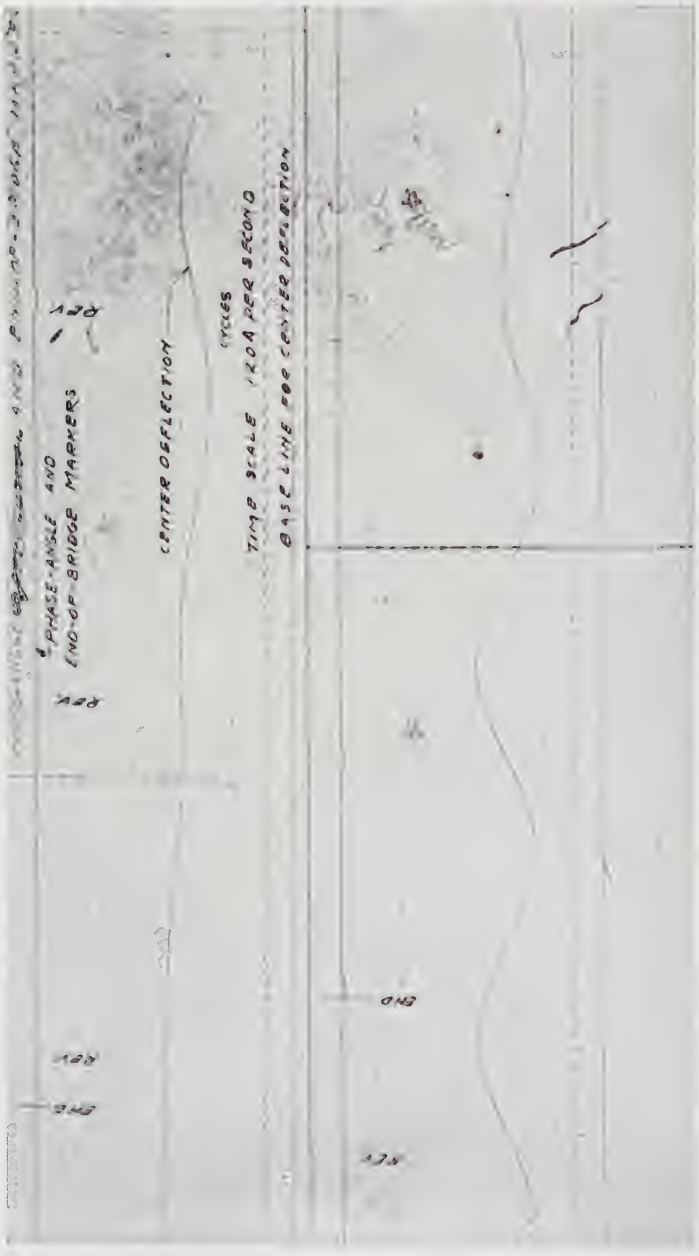


FIG. 17. SAMPLE OSCILLOGRAPH RECORD

center of the bridge, recorded directly, without reduction or multiplication, by a stylus attached to a rod at the center of the bridge. The third line from the top is a time scale formed by a reed which vibrated at 120 cycles per second and which was actuated by a solenoid excited by a 60-cycle alternating current. The bottom line is the fixed base line. The vertical intercepts between the bottom line and the second line from the top indicate the deflection of the center of the bridge. The deflection thus obtained does not include the dead-load deflection.

Damping was produced electrically with the apparatus shown in Fig. 15. Four aluminum plates were attached to the model of the bridge at the $\frac{1}{5}$ points and moved vertically in strong magnetic fields as the bridge deflected. Because the force required to move a conductor in a given magnetic field is proportional to the velocity at which the motion occurs, the vertical movement of the model of the bridge was resisted by a damping force that varied directly with the velocity of the bridge.

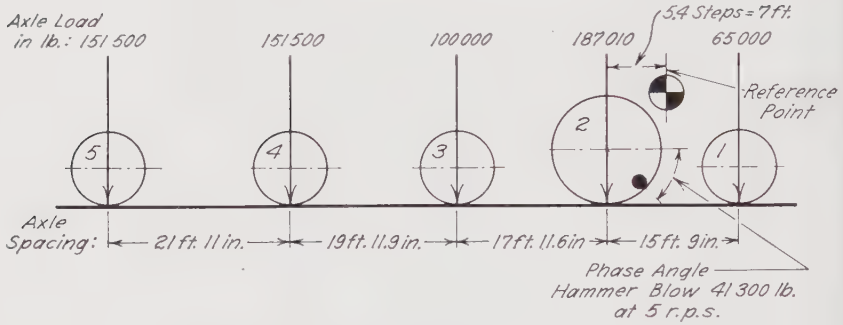
Five-Axle Model

The general procedure for testing the five-axle model and the data obtained were the same as for the single-axle model. The principles of similitude were observed in designing this model of the five-axle equivalent of locomotive J1-D. The axle loads and spacings of the prototype and model are shown on Fig. 18. It will be recalled that the five-axle prototype had one driver representing the three drivers of J1-D. The other wheels represented one each of the front and rear trucks of the locomotive and one each of the two trucks of the tender. Moreover, each wheel of the five-axle prototype represented the resultant of the corresponding group of wheels of J1-D in both magnitude and position. The hammer blow on the single driver of the five-axle prototype was 41 300 pounds at 5 r.p.s.*

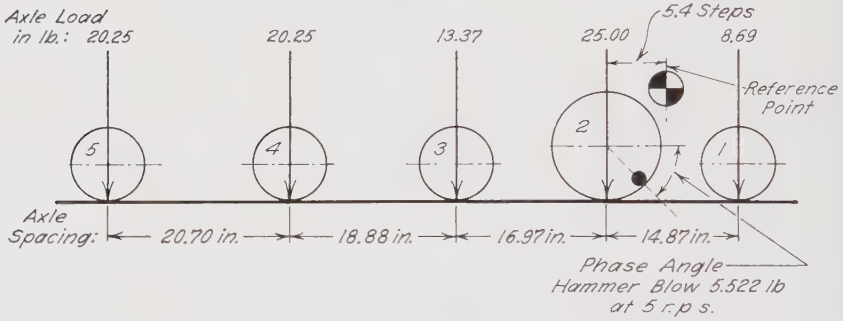
27. *Results of Tests of Single-Axle Models.*—The purpose of the tests of models was to compare the results of the tests with the results of a step-by-step analysis of the same models. Because the principles of similitude were observed in designing the models, the results of the analyses of the models are identical with the results of the analyses of the prototypes. In addition, the effect of certain variables was investigated with the experimental apparatus. It is sometimes more convenient to obtain the effect of variables over a range of values by experimental rather than by analytical methods.

The following tests were made with the model of the single-axle prototype equivalent of locomotive J1-D:

*See footnote, page 72.



(a)-Prototype – Five-Axle Equivalent to Locomotive J1-D



(b)-Model of Five-Axle Prototype Above

FIG. 18. PROTOTYPE AND MODEL OF FIVE-AXLE LOCOMOTIVE EQUIVALENT TO LOCOMOTIVE J1-D

(a) Tests at 5 r.p.s. with hammer blow, to compare the measured with the computed deflections

(b) Tests to determine the effect of the phase angle of the counterweight upon the center deflection

(c) Tests at 6.25 r.p.s. without hammer blow to determine speed effect

(d) Tests at velocities of 4.0, 4.4, and 5 r.p.s. with hammer blow to determine hammer blow plus speed effect.

The results of tests (a), the tests made to determine the deflection of the center of the bridge at a velocity of 5 r.p.s., are shown in Fig. 19. In these tests the reference point for the position of the single-axle model relative to the left end of the bridge is the axle itself. The phase angle of the counterweight was zero at the instant when the axle arrived at the left end of the bridge.

There are two curves at the top of Fig. 19. One shows the meas-

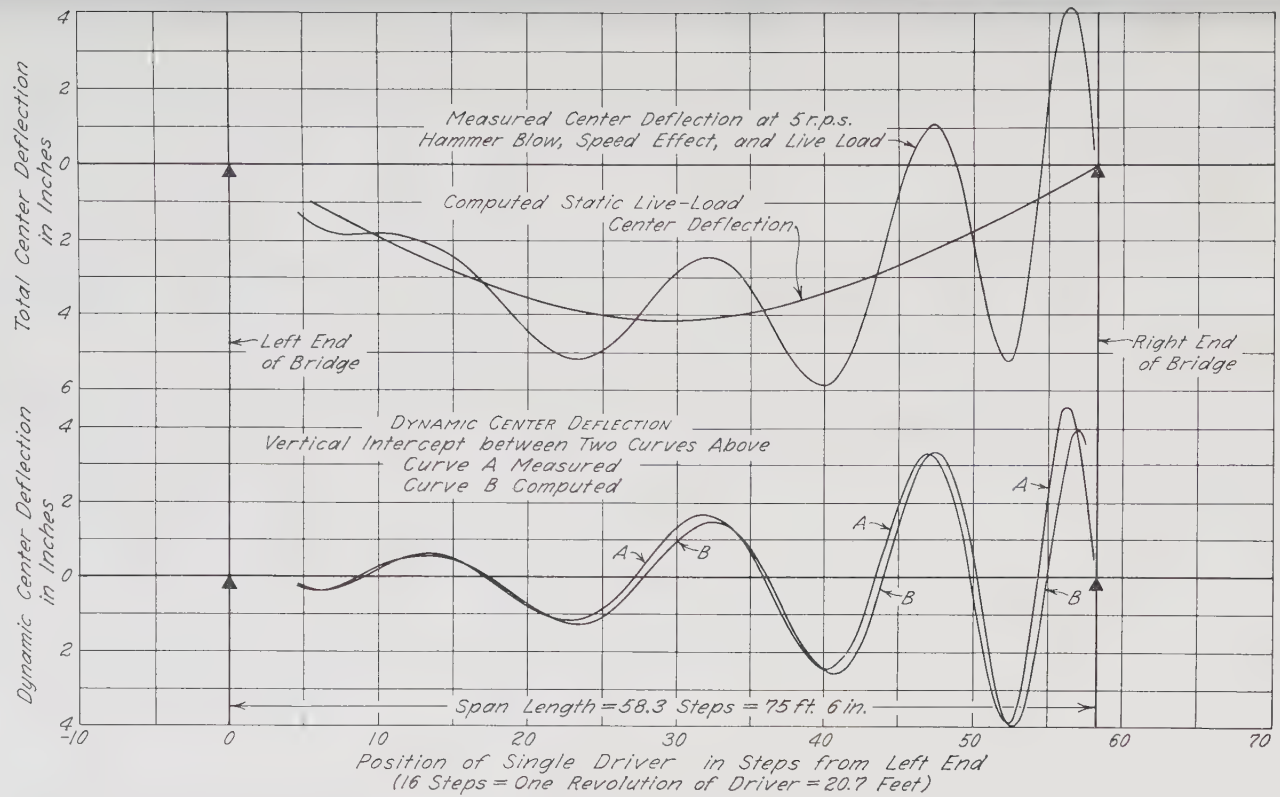


FIG. 19. MEASURED AND COMPUTED VALUES OF CENTER DEFLECTION OF BRIDGE 21 DUE TO HAMMER BLOW AND SPEED EFFECT; SINGLE-AXLE LOCOMOTIVE EQUIVALENT TO J1-D; MEASURED VALUES FROM MODEL TESTS; 5 R.P.S.; NO DAMPING

ured center deflection at a velocity of 5 r.p.s. obtained from the deflectometer record for a test of the model, and the other shows the computed static live-load center deflection. The vertical intercepts between the two curves, which represent the measured dynamic center deflection due to combined hammer blow and speed effect, is represented by curve A at the bottom of the figure. The computed dynamic center deflection due to combined hammer blow and speed effect, as determined by the step-by-step analysis, is represented by curve B, which is superimposed upon curve A. The correlation between these two curves indicates that the measured and computed values of the deflection at the center of the bridge agree very well. One interesting feature indicated by both diagrams is that the amplitude of the oscillations increased until the single wheel rolled off the bridge. That is, the maximum semiamplitude occurred when the static deflection was nearly zero. It should be noted, however, that this was for a single axle.

Tests (b), the tests made to determine the effect of the phase angle of the counterweight upon the center defection, were made with the model of the single-axle equivalent of locomotive J1-D. The phase angle of the counterweight was adjusted in four positions 90 degrees apart, the phase angle being determined at the instant when the axle arrived at the left end of the bridge. The positions of the counterweight just as the axle of the model arrived at the left end of the bridge are shown at the bottom of Fig. 20, and are designated as phase angles 0, 90, 180 and 270 deg. The phase angle is measured clockwise from a horizontal line through the axle directed toward the front end of the model. All tests were made at a speed of 5 r.p.s. and deflectometer records were made for each phase angle of the counterweight. The semiamplitude of the oscillations of the center of the bridge due to hammer blow plus speed effect was determined from the measured values of the deflectometer record and the computed static live-load deflection, and the maximum value of this semiamplitude was determined for each phase angle of the counterweight. These values were entered in column 7 of Table 2. The relation between the phase angle of the counterweight at the instant when the axle arrived at the left end of the bridge, and the maximum semiamplitude of the center deflection is shown by the upper curve of Fig. 20.

The maximum total center deflection due to a moving live load minus the maximum static live-load center deflection is equivalent to what is generally known as impact effect. This is usually written maximum (live load plus impact) — maximum (static live load) = im-

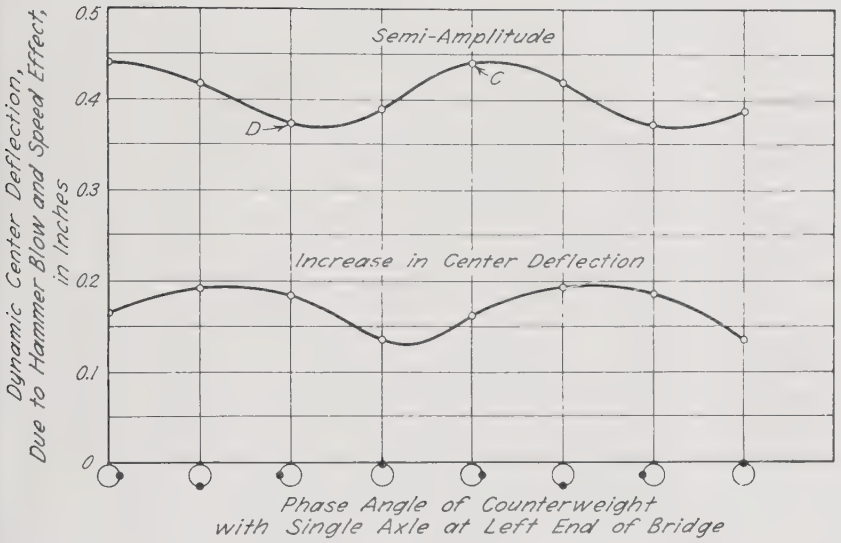


FIG. 20. EFFECT OF PHASE ANGLE OF COUNTERWEIGHT ON CENTER DEFLECTION TESTS OF MODEL OF SINGLE-AXLE EQUIVALENT TO J1-D AT 5 R.P.S.; NO DAMPING

pact effect. The curves of Fig. 19 show that the maximum center deflection due to a moving live load and the maximum static live-load center deflection of the bridge do not occur at the same position of the single-axle from the left end. Nevertheless, it is the custom of the profession to consider this quantity to be the impact effect. The impact, expressed as a percentage of the maximum static live-load center deflection, is

$$\frac{\text{Maximum (live load plus impact)} - \text{Maximum (static live load)}}{\text{Maximum (static live load)}} \times 100 .$$

Following this custom, the maximum center deflection (the deflection due to hammer blow plus speed effect and live load) which corresponds to the maximum (live load plus impact), has been entered in column 4 of Table 2, and designated deflection A. Deflection A minus the maximum static live-load center deflection has been entered in column 5 and designated deflection B. The impact percentage as described in the foregoing is equal to

$$\frac{\text{Deflection B}}{\text{Maximum (static live load)} = 0.419} \times 100$$

TABLE 2

INFLUENCE OF PHASE ANGLE OF COUNTERWEIGHT UPON CENTER DEFLECTION

Tests of model of single-axle equivalent of J1-D; 5 r.p.s.; no damping

Velocity of Model	Phase Angle of Counterweight at Left End of Bridge	Maximum Center Deflection				Maximum Semiamplitude	
		Portion of Span Traversed by Axle for Maximum	Deflection A*	Deflection B†	Deflection B‡ / 0.419	Semiamplitude	Semiamplitude‡ / 0.419
r.p.s.	deg.		in.	in.	per cent	in.	per cent
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
5.00	0	0.68	0.582	0.163	38.9	0.441	105.2
5.00	90	0.58	0.610	0.191	45.6	0.418	100.0
5.00	180	0.50	0.602	0.183	43.7	0.373	89.0
5.00	270	0.44	0.555	0.136	32.5	0.390	93.2

*Deflection A = maximum center deflection due to hammer blow, speed effect, and live load.
†Deflection B = increase in center deflection; deflection A minus maximum static live-load center deflection of 0.419 inches. This is the deflection corresponding to what is usually known as impact.
‡Percentage of maximum static live-load center deflection.

and this has been entered in column 6 of the same table. The impact effect computed in this manner will be referred to as the increase in center deflection. The relation between the phase angle of the counterweight when the axle arrived at the left end of the bridge and the deflection B in column 5, is shown by the lower curve of Fig. 20.

The ordinates of the curves of Fig. 20 represent the maximum values of the dynamic center deflections while the axle rolled across the bridge. The abscissas are the phase angles of the counterweight at the instant when the axle arrived at the left end of the bridge, as indicated at the bottom of the figure. Thus, point C shows that when the axle arrived at the left end with the counterweight at phase angle 0 deg., the maximum semiamplitude of the center deflection that occurred as the axle rolled across the bridge was 0.441 inches. In contrast with this, point D shows that when the same axle with the same counterweight arrived at the left end of the bridge at the same velocity but with the counterweight at phase angle 180 deg., the maximum semiamplitude that occurred as the axle rolled across the bridge was only 0.373 inches. The difference is due to the difference in the phase angle of the counterweight. The points representing the phase angles of the counterweight have been repeated in the figure to give the continuity of the curves. It is of interest to note that, for a single-axle load, the maximum center deflection depends to a considerable extent upon the phase angle of the counterweight.

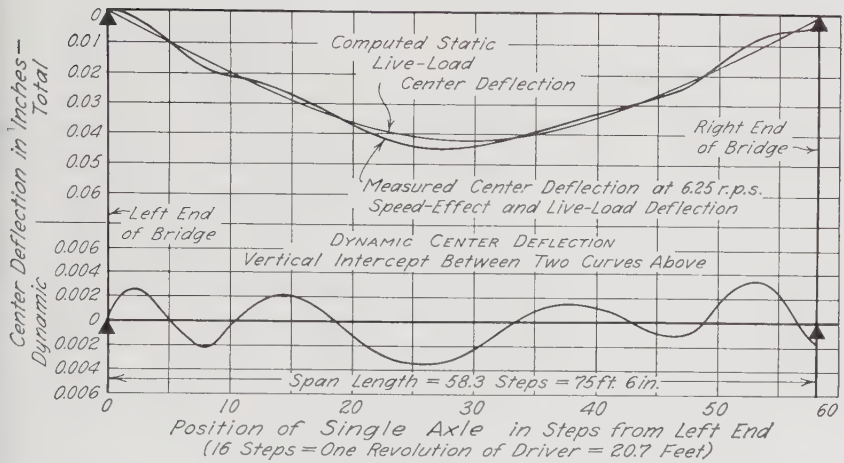


Fig. 21. Measured Center Deflection of Bridge 21 Due to Speed Effect; Single-Axle Locomotive J1-D Without Hammer Blow; Measured Values From Model Tests, 6.25 r.p.s.; No Damping

Tests (c), the tests designed to determine the speed effect without hammer blow, were made with a model similar to the model of the single-axle equivalent of locomotive J1-D except that there was no counterweight; i.e., with a smoothly rolling single wheel. The tests were made at a velocity of 6.25 r.p.s. and the results are given in Fig. 21. The experimentally-determined curve for the center deflection of the bridge at a velocity of 6.25 r.p.s., (88 m.p.h.) is shown at the top of the figure superimposed on a curve showing the computed static live-load center deflection. The vertical intercepts between the two curves, which are considered as representing the speed effect, are found at the bottom of the figure. The deflection due to speed effect, though small, indicates that oscillations developed as a result of speed even with a smoothly-rolling wheel. The presence of centrifugal effect may also be observed.

The results of tests (d), the tests at 4.0, 4.4, and 5.0 r.p.s., are given in Table 3. In these tests the phase angle of the counterweight was zero at the instant when the axle arrived at the left end of the bridge. Tests (b) show that this phase angle gives maximum semiamplitude, and a value of the increase in center deflection slightly smaller than the maximum. Tests at each velocity were made with each of three weights of the counterweight. The hammer blow at 5 r.p.s. for the three counterweights had values of 4.41, 5.52, and 6.62 pounds. The medium weight, with a hammer blow of 5.52 pounds at 5 r.p.s., cor-

TABLE 3

INFLUENCE OF VELOCITY AND WEIGHT OF COUNTERWEIGHT UPON CENTER DEFLECTION

Tests of model of single-axle equivalent of J1-D; no damping

Hammer Blow at 5 r.p.s.	Phase Angle of Counterweight With Axle at Left End of Bridge	Velocity of Model	Maximum Center Deflection			Maximum Semiamplitude	
			Deflection A*	Deflection B†	Deflection B‡ / 0.419	Semiamplitude	Semiamplitude‡ / 0.419
pounds	deg.	r.p.s.	in.	in.	per cent	in.	per cent
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
4.41	0	4.03	0.465	0.046	11.0	0.157	37.5
		4.40	0.482	0.063	15.0	0.198	47.2
		4.92	0.512	0.093	22.2	0.294	70.1
5.52	0	4.06	0.510	0.091	21.7	0.210	50.1
		4.42	0.515	0.096	22.9	0.292	69.7
		5.03	0.582	0.163	38.9	0.441	105.2
6.62	0	4.01	0.535	0.116	27.7	0.194	46.3
		4.40	0.562	0.143	34.1	0.320	76.3
		4.78	0.610	0.193	46.1	0.443	105.8

*Deflection A = maximum center deflection due to hammer blow, speed effect, and live load.
†Deflection B = increase in center deflection; deflection A minus maximum static live-load center deflection of 0.419 inches. This is the deflection corresponding to what is usually known as impact.
‡Percentage of maximum static live-load center deflection.

responds to that of the model of the single-axle equivalent of locomotive J1-D. The other two hammer blows were 20 per cent greater than and less than this medium value. The speed of 4.4 r.p.s. was the synchronous speed corresponding to the loaded frequency of the bridge when the single-axle load was at the center of the bridge.

There are three deflections given in Table 3 which are of particular interest: (1) column 4 contains the maximum center deflection of the bridge (the deflection due to hammer blow, speed effect and live load), designated as deflection A; (2) column 5 contains the increase in the center deflection, deflection A minus the maximum static live-load center deflection, which has been designated as deflection B; (3) column 7 contains the maximum semiamplitude of the oscillations which indicate the cumulative effect of the hammer blow plus speed effect.

At a velocity above 5 r.p.s. the single wheel with the heaviest counterweight tended to lose contact with the rail due to the large vertical acceleration. The deflections in Table 3 for this counterweight may not be the maximum values.

28. *Results of Tests of Five-Axle Models.*—The purpose of the model tests was stated at the beginning of Section 27. Because of the

close similarity between the results of the step-by-step analysis for locomotive J1-D and those for the five-axle equivalent, a considerable amount of experimental data was obtained with this model. From these data a better picture can be obtained of the effect of a locomotive on a bridge than has been possible up to the present.

The reference point for the position of the five-axle model from the left end of the bridge is a point 5.4 steps in front of the single driver, axle 2. The phase angle of the counterweight is given at the instant when this reference point arrived at the left end of the bridge.

The following tests were made with the five-axle model of the five-axle prototype equivalent to J1-D:

(a) Tests to determine the effect of the phase angle of the counterweight upon the center deflections

(b) Tests at 3.7, 5.0, 5.7 and 6.3 r.p.s. without hammer blow, with and without damping

(c) Tests at 3.7, 5.0, 5.7 and 6.3 r.p.s. with hammer blow, with and without damping

(d) Tests to determine the damping force.

The tests (a) were made to determine whether the phase angle was significant and, if so, to find the position of the counterweight for maximum effect to be used in subsequent tests. The phase angle of the counterweight was adjusted in four positions 90 degrees apart, the phase angle being determined at the instant when the reference point arrived at the left end of the bridge. Tests were made by rolling the model across the bridge with the counterweight adjusted to the four positions shown at the bottom of Fig. 22. Deflectometer records of the center deflection were made at each of the velocities 4.15, 5.0, 5.72 and 6.33 r.p.s.

The results of the tests, measured center deflections from the experimental records, are given in Table 4. Column 1 gives the velocity of the model in r.p.s., column 2, the phase angle of the counterweight at the instant when the reference point arrived at the left end of the bridge.

Columns 3, 4, 5, and 6 contain data relative to the maximum center deflection of the bridge and the computation of the increase in center deflection. Column 3 indicates the portion of the span traversed by the reference point of the model when the center deflection was a maximum. Column 4 contains the maximum center deflection A (the deflection due to hammer blow plus speed effect and live load). Column 5 contains the value of the increase in center deflection, deflection A

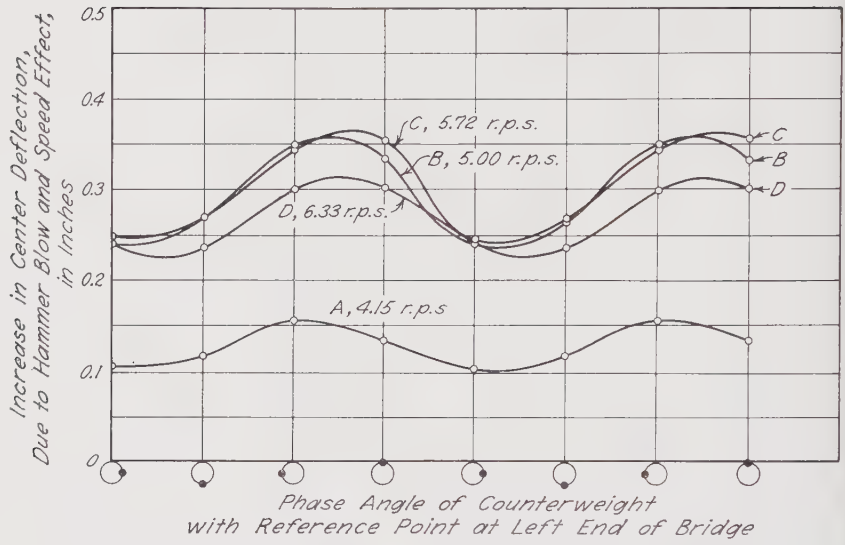


Fig. 22. Effect of Phase Angle of Counterweight on Center Deflection Tests of Model of Five-Axle Locomotive Equivalent to J1-D at Various Velocities Without Damping

minus the maximum static live-load center deflection, which corresponds to maximum (live load plus impact) — maximum (static live load). This has been designated as deflection B. Column 6 contains the value of the ratio

$$\frac{\text{Deflection B}}{\text{Maximum (static live load)}} \times 100,$$

usually known as the impact effect. Columns 7, 8, and 9 contain data relative to the maximum semiamplitude of the oscillation. Column 7 indicates the portion of the span traversed by the reference point when the semiamplitude of the center deflection was greatest; column 8 indicates the maximum semiamplitude; and column 9 indicates the ratio of the maximum semiamplitude to the maximum static live-load center deflection.

The effect of the phase angle of the counterweight upon the center deflection of the bridge is shown by the curves of Fig. 22. The ordinates of these curves represent the increase in center deflection (deflection A minus the maximum static live-load center deflection), this quantity being designated as deflection B in column 5 of Table 4. The

Table 4

Influence of Phase Angle of Counterweight Upon Center Deflection at Various Velocities

Tests of model of five-axle equivalent of J1-D; no damping

Velocity of Model	Phase Angle of Counterweight With Reference Point at Left End of Bridge	Maximum Center Deflection				Maximum Semiamplitude		
		Portion of Span Traversed by Reference Point for Maximum	Deflection A*	Deflection B†	Deflection B‡ / 0.419	Portion of Span Traversed by Reference Point for Maximum	Semi-amplitude	Semiamplitude‡ / 0.419
r.p.s.	deg.				per cent		in.	per cent
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
4.15	0	0.88	0.521	0.102	24.4	0.82	0.132	31.5
	90	0.82	0.544	0.126	27.4	0.82	0.133	31.8
	180	0.77	0.575	0.156	37.2	0.77	0.157	37.5
	270	0.72	0.552	0.133	31.8	0.72	0.136	32.4
5.00	0	0.96	0.658	0.239	57.0	0.96	0.261	62.3
	90	0.87	0.680	0.261	62.3	1.14	0.287	68.5
	180	1.08	0.768	0.349	83.0	1.08	0.398	95.0
	270	1.03	0.752	0.333	79.5	1.03	0.364	86.9
5.72	0	1.02	0.664	0.245	58.5	1.02	0.274	65.4
	90	0.92	0.687	0.268	64.0	0.93	0.293	170.0
	180	0.86	0.760	0.341	81.5	1.00	0.374	89.3
	270	1.09	0.775	0.356	85.0	1.09	0.399	95.3
6.33	0	0.74	0.666	0.247	59.0	0.74	0.247	59.0
	90	0.96	0.650	0.231	55.1	0.96	0.254	60.6
	180	0.90	0.716	0.297	70.9	0.90	0.328	78.3
	270	0.82	0.721	0.302	72.1	0.82	0.308	73.5

*Deflection A = maximum center deflection due to hammer blow, speed effect and live load.

†Deflection B = increase in center deflection; deflection A minus maximum static live-load center deflection of 0.419 inches. This is the deflection corresponding to what is usually known as impact.

‡Percentage of maximum static live-load center deflection.

abscissas represent the phase angle of the counterweight at the instant when the reference point arrived at the left end of the bridge.

Additional information on the effect of the phase angle is given in the following paragraph where the difference between the largest and smallest dynamic deflection at any one velocity is shown. This difference is due in part to the phase angle of the counterweight, and in part to the speed effect. The oscillations due to speed effect are not necessarily in phase with the oscillations due to the hammer blow. Therefore, at some points the speed effect may act with the hammer blow while at other points it may act against the hammer blow.

The differences below are computed with and without speed effect.

With Speed Effect

(1) Increase in center deflection; dynamic deflection obtained from the difference between the maximum center deflection due to a moving live load (the deflection due to hammer blow, speed effect and live load), and the maximum static live-load center deflection. These deflections may be found in column 5 of Table 4.

Velocity	Phase Angle deg.	Increase in Center Deflection, in.	Difference in.
5 r.p.s.	180	Largest 0.349	
5 r.p.s.	0	Smallest 0.239	0.110
5.7 r.p.s.	270	Largest 0.356	
5.7 r.p.s.	0	Smallest 0.245	0.111

(2) Maximum semiamplitude; dynamic deflection obtained from the difference between the center deflection due to a moving live load (the deflection due to hammer blow, speed effect and live load), and the static live-load center deflection with the model at the same distance from the left end of the bridge. These deflections are shown in column 8 of Table 4.

Velocity	Phase Angle deg.	Maximum Semiamplitude in.	Difference in.
5 r.p.s.	180	Largest 0.398	
5 r.p.s.	0	Smallest 0.261	0.137

Without Speed Effect

(1) Increase in center deflection; dynamic deflection obtained from the difference between the maximum center deflection due to a moving live load with hammer blow (the deflection due to hammer blow, speed effect and live load), and the center deflection due to a moving live load without hammer blow (the deflection due to speed effect plus live load) with the model at the same distance from the left end of the bridge.

Velocity	Phase Angle deg.	Increase in Center Deflection, in.	Difference in.
6.3 r.p.s.	270	Largest 0.263	
6.3 r.p.s.	180	Smallest 0.224	0.039

(2) Maximum semiamplitude; dynamic deflection obtained from the difference between the center deflection due to a moving live load with hammer blow (the deflection due to hammer blow, speed effect and live load), and the center deflection due to a moving live load without hammer blow (the deflection due to speed effect plus live load) with the model at the same distance from the left end of the bridge.

Velocity	Phase Angle deg.	Maximum Semiamplitude in.	Difference in.
5.7 r.p.s.	90	Largest 0.341	
5.7 r.p.s.	270	Smallest 0.296	0.045

From these tests the phase angle of the counterweight for the rest of the tests was chosen. For velocities of 5 r.p.s. or less a phase angle of 180 deg. was used, and for those above 5 r.p.s., a phase angle of 270 deg. These were the counterweight positions for maximum increase in center deflection.

Tests (b) were made on the five-axle model without counterweight to determine the deflection produced by five smoothly rolling wheels. Except for the lack of a counterweight, the model was identical with the model of the five-axle equivalent of locomotive J1-D which was used in the tests just described. The speed effect was obtained by sub-

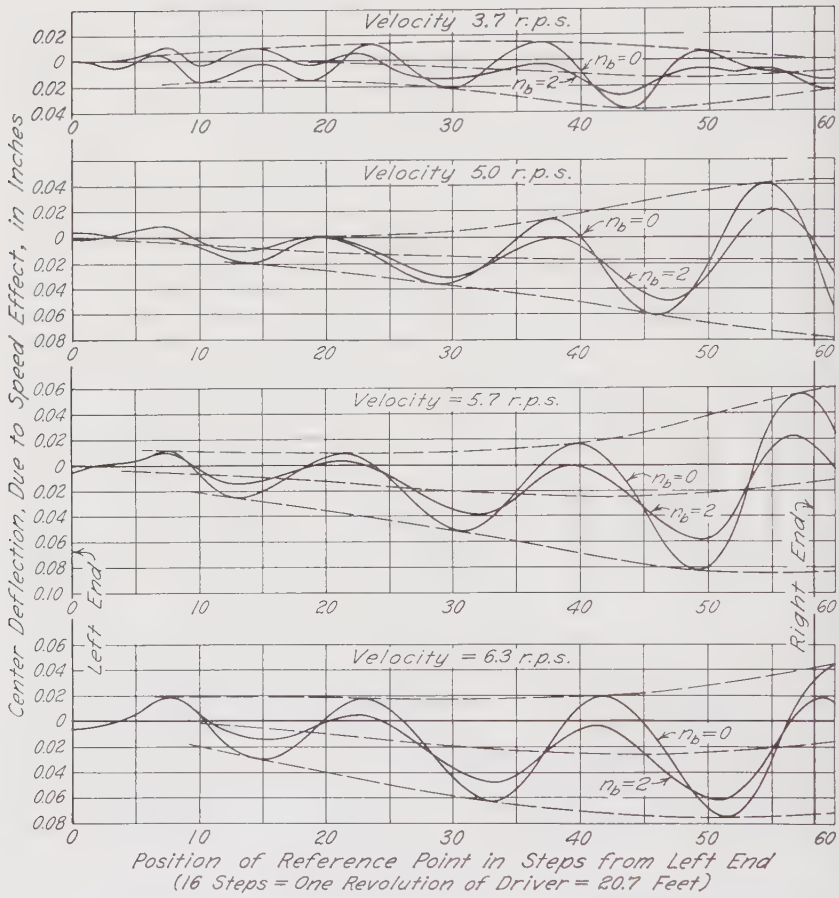


FIG. 23. MEASURED CENTER DEFLECTION DUE TO SPEED EFFECT; FIVE-AXLE MODEL WITHOUT COUNTERWEIGHT TESTED AT VARIOUS VELOCITIES, WITH AND WITHOUT DAMPING

tracting the computed static live-load center deflection from the measured center deflection taken from the deflectometer records. The resulting curves showing the speed effect are given in Fig. 23. Two sets of tests were made at each of four speeds, 3.7, 5.0, 5.7 and 6.3 r.p.s. For one set of tests there was no damping ($n_b = 0$) and for the other set there was a damping which varied with the vertical velocity of the bridge, and which was represented in magnitude by the expression $n_b = 2$.* The two speed-effect curves for each of the four speeds, one

*This is explained on pages 47 and 99.

with damping and the other without damping, are superimposed upon each other in the figure.

Two dash lines were drawn tangent to the peaks of the curves, $n_b = 0$, and an average dash line was drawn between them. The ordinates of the average dash line represent the deflection due to centrifugal effect. The oscillations were caused by the series of wheels which came onto the bridge quickly and imparted a series of impulses to it. The centrifugal effect and the impulse effect both increased with the velocity.

Figure 28 shows a comparison between the measured and the computed speed effect. Curve A shows the measured center deflection obtained from the deflectometer record of tests of the five-axle model, and curve B the computed deflections from a step-by-step analysis of the prototype of this model, which, because of similitude, will be the same as an analysis of the model. These curves show close agreement between measured and computed center deflections for speed effect.

Tests (c) were made with the five-axle model using three different counterweights. The model was the same as the model of the five-axle equivalent of locomotive J1-D, except for the weight of the counterweight. The variables were velocity and hammer blow. There was no bridge damping in these tests. Tests were made at approximately each of the velocities 3.7, 5.0, 5.7 and 6.3 r.p.s. with counterweights which would give hammer blows at 5.0 r.p.s. of 0.0, 4.41, 5.52 and 6.61 pounds. The hammer blow of 5.52 pounds at 5 r.p.s. is the same as the hammer blow of the model of the five-axle equivalent of J1-D. The other hammer blows are 20 per cent greater and 20 per cent less, respectively. The object of the tests was to determine the effect of each of the factors, velocity and hammer blow, upon the dynamic center deflection of the bridge.

Deflectometer records were made for each test and the various deflections were obtained from these curves in the manner already explained. These deflections are given in Table 5. The relation between the velocity in r.p.s. and the increase in center deflection, deflection B, column 6 of Table 5, is given by the curves of Fig. 24. There are four curves, and each shows the increase in center deflection for a different weight of the counterweight. The largest value of the increase in center deflection is at a velocity slightly greater than 5 r.p.s. for all weights of the counterweight.

Tests were also made with the model with a counterweight producing a hammer blow of 5.52 pounds at 5 r.p.s., at speeds of 3.7, 5.0,

Table 5

Influence of Velocity and Weight of Counterweight Upon Center Deflection

Tests of model of five-axle equivalent of J1-D; no damping

Hammer Blow at 5 r.p.s.	Phase Angle of Counterweight With Reference Point at Left End of Bridge deg.	Velocity of Model r.p.s.	Maximum Center Deflection				Maximum Semiamplitude		
			Portion of Span Traversed by Reference Point for Maximum	Deflection A* in.	Deflection B† in.	Deflection B‡ / 0.419 per cent	Portion of Span Traversed by Reference Point for Maximum	Semi-amplitude in.	Semiamplitude‡ / 0.419 per cent
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
4.41	180	3.67	0.73	0.498	0.079	18.9	0.73	0.080	19.1
	180	5.00	1.07	0.693	0.274	65.5	1.07	0.319	76.2
	270	5.77	1.09	0.702	0.283	67.6	1.09	0.334	79.8
	270	6.33	0.82	0.661	0.242	57.8	0.82	0.249	59.5
5.52	180	3.68	0.74	0.518	0.099	23.6	0.74	0.100	23.9
	180	5.00	1.08	0.768	0.349	83.3	1.08	0.398	95.0
	270	5.74	1.09	0.775	0.356	85.0	1.09	0.399	95.3
	270	6.33	0.82	0.721	0.302	72.1	0.82	0.308	73.5
6.61	180	3.74	0.76	0.538	0.119	28.4	0.76	0.119	28.4
	180	5.00	1.09	0.834	0.415	99.1	1.09	0.467	112.0
	270	5.72	1.10	0.824	0.405	96.8	1.10	0.459	109.7
	270	6.30	0.82	0.750	0.331	79.0	1.15	0.354	84.6

*Deflection A = maximum center deflection due to hammer blow, speed effect, and live load.

†Deflection B = increase in center deflection; deflection A minus maximum static live-load center deflection of 0.419 inches. This is the deflection corresponding to what is usually known as impact.

‡Percentage of maximum static live load center deflection.

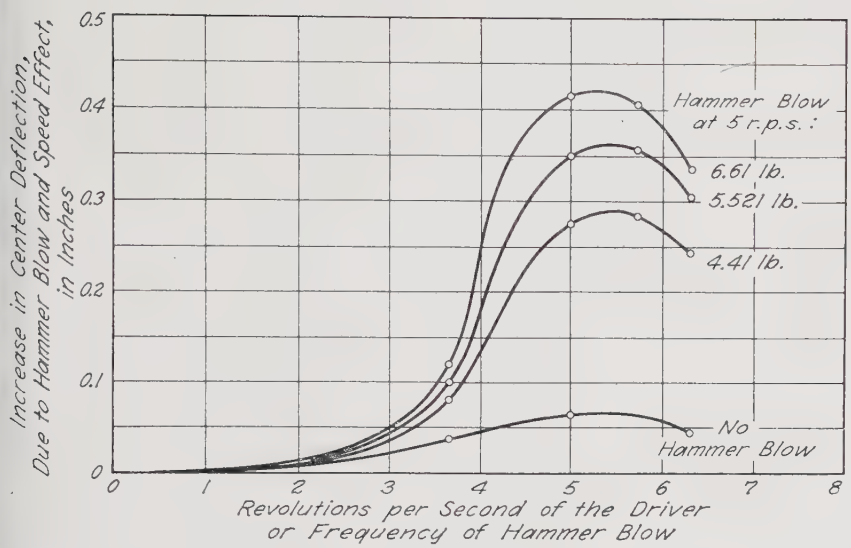


Fig. 24. Relation Between Velocity and Increase in Center Deflection Due to Hammer Blow and Speed Effect; Tests of Five-Axle Model With Several Different Hammer-Blow Forces; No Damping

5.7 and 6.3 r.p.s. with each of four different magnitudes of the damping force. The increase in center deflection due to combined hammer blow and speed effect, obtained from the deflectometer records, is given in Table 6. The relation between the velocity of the model and the increase in center deflection, deflection B, due to combined hammer blow and speed effect, is shown by Fig. 25, a separate curve being drawn for each magnitude of damping. It is noteworthy that the largest value of the increase in center deflection due to combined hammer blow and speed effect occurred at a velocity slightly greater than 5 r.p.s. for all degrees of damping, and that the deflection decreased with an increase in the damping coefficient, but at a decreasing rate.

In order to obtain data on the center deflection due to hammer blow alone, deflection records were made without a counterweight for each of these velocities and dampings, using the same five-axle model. The speed-effect plus live-load deflection was measured on these records for the position of the model which gave the maximum center deflection due to combined hammer blow, speed effect and live load for

TABLE 6

INFLUENCE OF VELOCITY AND MAGNITUDE OF DAMPING UPON CENTER DEFLECTION

Tests of model of five-axle equivalent of J1-D

Damping Coefficient		Revolutions per Second of the Driver											
		3.68			5.00			5.74			6.33		
Δ^*	n_b^*	Portion of Span Traversed by Reference Point for Maximum	Deflection B†	Deflection B† / 0.119	Portion of Span Traversed by Reference Point for Maximum	Deflection B†	Deflection B† / 0.419	Portion of Span Traversed by Reference Point for Maximum	Deflection B†	Deflection B† / 0.419	Portion of Span Traversed by Reference Point for Maximum	Deflection B†	Deflection B† / 0.419
			in.	per cent		in.	per cent		in.	per cent		in.	per cent
		(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
0	0	0.74	0.099	23.6	1.08	0.349	83.3	1.09	0.356	85.0	0.82	0.302	72.1
0.0327	0.396	0.75	0.093	22.2	1.08	0.290	69.2	0.78	0.300	71.6	0.82	0.275	65.6
0.0749	0.906	0.76	0.092	22.0	0.80	0.249	59.5	0.78	0.268	64.0	0.81	0.243	58.0
0.119	1.44	0.76	0.087	22.8	0.81	0.220	52.5	0.78	0.237	56.6	0.80	0.226	54.0
0.165	2.00	0.76	0.081	19.3	0.82	0.202	48.2	0.78	0.220	52.5	0.81	0.207	49.4

*See page 99.
†See footnotes of Table 5.

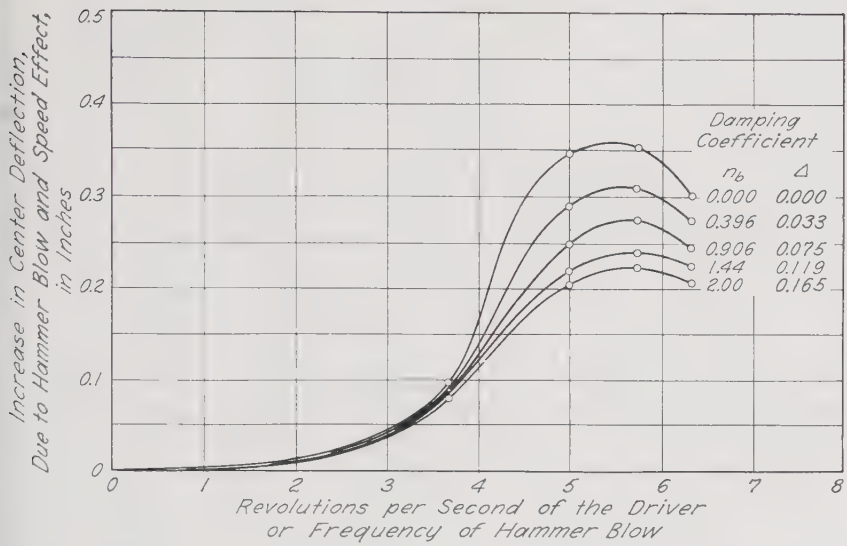


FIG. 25. RELATION BETWEEN VELOCITY AND INCREASE IN CENTER DEFLECTION DUE TO HAMMER BLOW AND SPEED EFFECT; TESTS OF MODEL OF FIVE-AXLE EQUIVALENT TO LOCOMOTIVE J1-D; DAMPING VARIED

the counterweight model. The difference between these two deflections gives the deflection due to hammer blow alone. The data obtained in this manner are given in Table 7. The relation between the velocity of the model and the increase in center deflection due to hammer blow alone is given in Fig. 26, a curve being drawn for each magnitude of damping. It is significant that the greatest increase in center deflection due to hammer blow alone occurred at a velocity slightly less than 6 r.p.s., a small increase over the velocity which produced the greatest increase in center deflection due to combined hammer blow and speed effect. These deflections decreased with an increase in the damping coefficient, but at a decreasing rate.

It is possible for the maximum center deflection due to hammer blow plus live load, to be greater at a position of the model other than the one at which the maximum due to the combined hammer blow, speed effect and live load occurs. This was observed on only one record, for a case in which the velocity was 5 r.p.s. and the damping $n_b = 0.40$.

Table 7

Influence of Velocity and Magnitude of Damping Upon Center Deflection Due to Hammer Blow Alone

Tests of model of five-axle equivalent of J1-D

Damping Coefficient		Revolutions per Second of the Driver									
		3.68					5.00				
Δ	n_b	Portion of Span Traversed by Reference Point for Maximum	Deflection A* in.	Deflection D† in.	Deflection C‡ in.	$\frac{\text{Deflection C¶}}{0.419}$ per cent	Portion of Span Traversed by Reference Point for Maximum	Deflection A in.	Deflection D in.	Deflection C in.	$\frac{\text{Deflection C}}{0.419}$ per cent
		(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
0	0	0.74	0.518	0.454	0.064	15.3	1.08	0.768	0.458	0.310	74.0
0.033	0.40	0.75	0.512	0.448	0.064	15.3	0.81	0.697	0.461	0.236	56.3
0.075	0.91	0.76	0.511	0.447	0.064	15.3	0.80	0.668	0.465	0.203	48.5
0.119	1.44	0.76	0.506	0.445	0.061	14.6	0.81	0.639	0.458	0.181	43.2
0.165	2.00	0.76	0.500	0.441	0.059	14.1	0.82	0.621	0.457	0.164	39.1

*Deflection A = maximum center deflection due to hammer blow, speed effect, and live load.
†Deflection D = center deflection due to speed effect and live load with model in same position as for A; obtained from a separate test with no hammer blow.
‡Deflection C = center deflection due to hammer blow alone; deflection A minus deflection D.
¶Percentage of maximum static live-load center deflection.

TABLE 7 (CONCLUDED)

INFLUENCE OF VELOCITY AND MAGNITUDE OF DAMPING UPON CENTER DEFLECTION DUE TO HAMMER BLOW ALONE

Tests of model of five-axle equivalent of J1-D

Damping Coefficient		Revolutions per Second of the Driver									
		5.74					6.33				
Δ	n_b	Portion of Span Traversed by Reference Point for Maximum	Deflection A* in.	Deflection D† in.	Deflection C‡ in.	Deflection C¶ / 0.419 per cent	Portion of Span Traversed by Reference Point for Maximum	Deflection A in.	Deflection D in.	Deflection C in.	Deflection C / 0.419 per cent
		(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
0	0	1.09	0.775	0.434	0.341	81.3	0.82	0.721	0.449	0.272	64.9
0.033	0.40	0.78	0.719	0.455	0.264	63.0	0.82	0.694	0.457	0.237	56.5
0.075	0.91	0.78	0.687	0.459	0.228	54.4	0.81	0.662	0.452	0.210	50.1
0.119	1.44	0.78	0.656	0.453	0.203	48.4	0.80	0.645	0.458	0.187	44.6
0.165	2.00	0.78	0.639	0.455	0.184	43.9	0.81	0.626	0.456	0.170	40.6

*Deflection A = maximum center deflection due to hammer blow, speed effect, and live load.
†Deflection D = center deflection due to speed effect and live load with model in same position as for A; obtained from a separate test with no hammer blow.
‡Deflection C = center deflection due to hammer blow alone; deflection A minus deflection D.
¶Percentage of maximum static live-load center deflection.

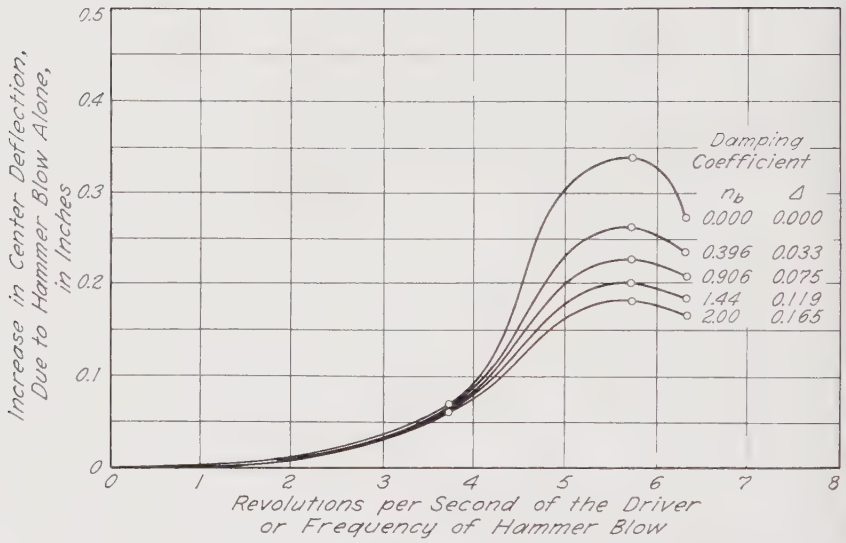


FIG. 26. RELATION BETWEEN VELOCITY AND INCREASE IN CENTER DEFLECTION DUE TO HAMMER BLOW ALONE; TESTS OF MODEL OF FIVE-AXLE EQUIVALENT TO LOCOMOTIVE J1-D; DAMPING VARIED

The data were as follows:

Portion of Span Traversed by Reference Point for Maximum Center Deflection	Center Deflection, inches				
	A	B	C	D	E
1.08	0.709	0.439	0.270	0.371	0.641
0.81	0.697	0.461	0.236	0.413	0.649

A = maximum center deflection, hammer blow, speed effect and live load.
B = center deflection, speed effect plus live load with model in same position as for A.
C = center deflection due to hammer blow, A − B.
D = static live-load center deflection with model in same position as for A.
E = maximum center deflection due to hammer blow plus live load, C+D.

It is interesting to note that the maximum deflection, E, occurred with the reference point of the model 0.81 of the span from the left end, whereas the maximum value of A occurred after the reference point had passed off the bridge.

Tests (d) were made to determine the magnitude of the damping force. They were made after each set of deflection tests of the model,

for each amount of damping used. The tests were made by causing the model bridge to vibrate with a load of about 35 pounds at the center of the span and making a deflectometer record of the center oscillations as they died down. The oscillations were started by hand. The weight was placed at the center in order that a large amplitude of oscillations could be obtained without causing the ends of the bridge to be lifted from the supports. The semiamplitudes of four successive waves were measured on each record, and designated as A, B, C, and D.

As explained in Section 26, the damping was produced by four aluminum plates attached to the bridge at the one-fifth points in such a manner that they moved up and down in a magnetic field as the bridge oscillated. The strength of the field was adjusted to give any predetermined magnitude of damping by changing the amperage in the circuit that excited the field. Since the resistance to the movement of a conductor through a field is proportional to the velocity, the magnitude of the damping varied with the vertical velocity of the bridge. The damping in the theory of Inglis* was considered to be a distributed force resisting the vertical motion of the bridge and directly proportional to the velocity of the motion at each point. This theory gives the expression for the distributed damping force as

$$\text{Damping force} = -4\pi n_b m \frac{dy}{dt}$$

where

m = mass per unit length of the bridge
y = the deflection of any point a distance x, from the left end
t = time
n_b = damping coefficient.

The formula for the ratio at which the amplitude of free vibration diminishes after every cycle is

$$R = e^{-2\pi \frac{n_b}{n_o}},$$

in which

R = ratio of successive semiamplitudes
n_b = damping coefficient
n_o = unloaded frequency of bridge.

*A Mathematical Treatise on Vibration of Railway Bridges, pages 16 and 17. C. E. Inglis, Cambridge University Press.

An analysis of the same bridge with the same damping force with an additional weight at the center gives the expression

$$R = e^{\left(-2\pi\frac{n_b}{n_o}\right)\frac{n_o'}{n_o}} \tag{11}$$

for the ratio, where n_o' = frequency of bridge with weight at middle. The values of n_b were derived from the experimental data by the use of Equation (11), using $n_o = 11.00$ cycles per second, and $n_o' = 5.06$ cycles per second.

The damping coefficient Δ is the one used in the A.R.E.A. 1936 report by Hunley. This coefficient is related to n_b by the expression

$$\Delta = \frac{2n_b n_2}{(n_o)^2},$$

where n_2 = frequency of bridge loaded with locomotive. For the five-axle equivalent locomotive and bridge 21, $n_2 = 5.00$ cycles per second, and $n_o = 11.00$ cycles per second; for these values, $\Delta = 0.0826 n_b$.

In the tests, four damping forces were used, one at each one-fifth point, and each force was proportional to the velocity of the bridge at the point where the force was applied. The moment curves for the computed and the measured damping forces agree closely, and the two deflection curves would almost coincide.

The expression for the distributed damping force given in the foregoing, when transformed into the nomenclature of this bulletin, becomes

$$\text{Damping force} = -4\pi n_b \frac{w}{g} v \sin\frac{\pi x_B}{L}. \tag{12}$$

By substituting the values of w for prototype and model, the damping force $= -704 n_b v \sin\frac{\pi x_B}{L}$ pounds per foot for the prototype, and $1.19 n_b v \sin\frac{\pi x_B}{L}$ pounds per foot for the model.

The substitution for n_b and v of their maximum values obtained in these tests of 2 cycles per second and 0.833 feet per second gives, for the prototype, distributed damping force $= 1172 \sin\frac{\pi x_B}{L}$ pounds per foot, and for the model, distributed damping force $= 1.97 \sin\frac{\pi x_B}{L}$

TABLE 8
EXPERIMENTAL DATA FOR DETERMINING DAMPING COEFFICIENT;
TESTS OF MODELS

Voltage	Record Number	Ratios of Successive Semi-amplitudes A, B, C, D		Average Ratios for Each Voltage	Values of Coefficient	
		Ratio A:C	Ratio B:D		n_b	Δ
(1)	(2)	(3)	(4)	(5)	(6)	(7)
18	1–1	0.901	0.897	0.900	0.396	0.033
	1–2	0.904	0.899			
	5–1	0.906	0.897			
	5–2	0.899	0.896			
36	9–1	0.791	0.786	0.787	0.906	0.075
	9–2	0.791	0.781			
	16–1	0.769	0.774			
	16–2	0.804	0.796			
54	17–1	0.699	0.683	0.683	1.44	0.119
	17–2	0.683	0.677			
	24–1	0.694	0.675			
	24–2	0.673	0.676			
72	25–1	0.589	0.596	0.589	2.00	0.165
	25–2					
	32–1	0.598	0.592			
	32–2	0.577	0.581			

pounds per foot. The friction at the ends of the model bridge was reduced to a minimum by means of roller bearings, so that it was negligible when compared with the amount of damping used in the tests.

Deflectometer records of the induced vibration of the bridge were made in duplicate before and after the impact tests of the model at each amount of damping, making a total of four records at each amperage. The ratios of successive semiamplitudes are given in columns (3) and (4) of Table 8 for the four records at each amperage, and the averages of the ratios of the successive semiamplitudes for a given amperage are given in column 5.

IX. RESULTS OF INVESTIGATION

29. *Introduction.*—The object of this investigation, as indicated in Chapter I, Section 1, was to examine the literature on the subject, to study critically certain features of the literature, and to develop ways and means for further investigation by analytical and experimental methods. The literature was discussed in Chapters II, III, and IV, and the scope of this bulletin was stated in Chapter V. The material selected for critical study is as follows:

(1) The use of a sine curve for the deflection curve of a bridge in the analysis for center oscillations.

(2) The representation of a locomotive by a single-axle load in the analysis of bridge oscillations.

(3) The conclusion that the center oscillation due to a locomotive hammer blow may be computed by the ordinary theory for the forced vibration of a spring with velocity damping. This spring theory was used in the British 1928 report, and in the A.R.E.A. 1936 report, for the purpose of deriving the impact allowance from field test data. The factors involved are as follows:

(a) The hammer blow force is applied a limited number of times for the locomotive, while the spring theory is based on an infinite number of hammer blows.

(b) The natural frequency of the bridge is changed by the weight of the locomotive and varies continuously during the passage of the locomotive, while in the spring theory the frequency remains constant.

(c) The difference between the hammer-blow effect on the center deflection when applied over the full length of the bridge, as occurs during the passage of the locomotive; and when applied continuously with full effectiveness at the center position, which is the force-deflection relationship used in the spring theory.

(4) The effect of a smoothly-rolling load on bridge deflections (speed effect).

(5) The effect of the phase angle of the hammer blow when the locomotive comes onto the bridge.

(6) Relationship between results obtained by the step-by-step analysis and by the tests of models.

(7) Comparison of the results obtained with the step-by-step analysis and by tests of models with the results obtained by other investigations.

Each of these items will be discussed in turn with reference to the analytical and experimental data in Chapters VII and VIII.

30. *Use of Sine Curve as Deflection Curve of Loaded Bridge.*— The deflection curve of a bridge was assumed to be a sine curve for the purpose of simplifying the analysis. This assumption determines the deflection, velocity, and acceleration relationship between different points of the bridge. In order to check this assumption several comparisons were made between the measured and the computed values of the center deflection of a bridge due to rapidly moving loads with and without hammer blow, the measured values being taken from records of laboratory tests of models in each instance.

A comparison of the measured and the computed values of the center deflection due to a single-axle load is given in Fig. 19, page 79. This is probably the most severe test, since the deflection curve for a single load will have the greatest deviation from the sine curve. The large weight of the wheel, relative to the weight of the bridge, would likewise magnify errors in acceleration because of the large concentrated mass-acceleration effect of the wheel. The curves of Fig. 19 show good agreement between the computed and the measured values of the deflections except at the right end where the wheel goes off the bridge.

A similar comparison for a series of five axles with hammer blow is shown in Fig. 27, page 104. As would be expected, the agreement between the measured and the computed values was much closer for the five-axle model than for the single-axle model.

A comparison between the measured and the computed values of the center deflection due to five smoothly-rolling loads without hammer blow is shown in Fig. 28, page 105.

The centrifugal effect and the oscillations due to the impulses of the individual wheels as they come onto the bridge are evident in both the measured and the computed deflections. The agreement between the measured and the computed deflections, considering relative values, is not as close in this case as in the previous comparison of hammer-blow effects. It should be noted, however, that the deflection due to speed effect is only about 0.20 as great as the deflection due to hammer-blow effect. The amplitude of the oscillation agrees very well, while the measured centrifugal effect appears to be greater than the computed centrifugal effect. The maximum measured centrifugal deflection is approximately 0.062 inches and the maximum computed deflection is 0.054 inches.

The foregoing evidence indicates that the sine-curve assumption is accurate enough for the computation of the dynamic effect of a series of wheels with or without hammer blow as used in the step-by-step analysis.

31. *Representation of Locomotive by Single-Axle Equivalent Locomotive.*—The representation of the locomotive by a single-axle load was made necessary by the method of analysis of the British 1928 report. The step-by-step analysis makes it possible to introduce a series of axle loads with the spacings and weights of the locomotive axles. A study was made to determine whether or not the representation of a locomotive by a single-axle load introduced errors too large

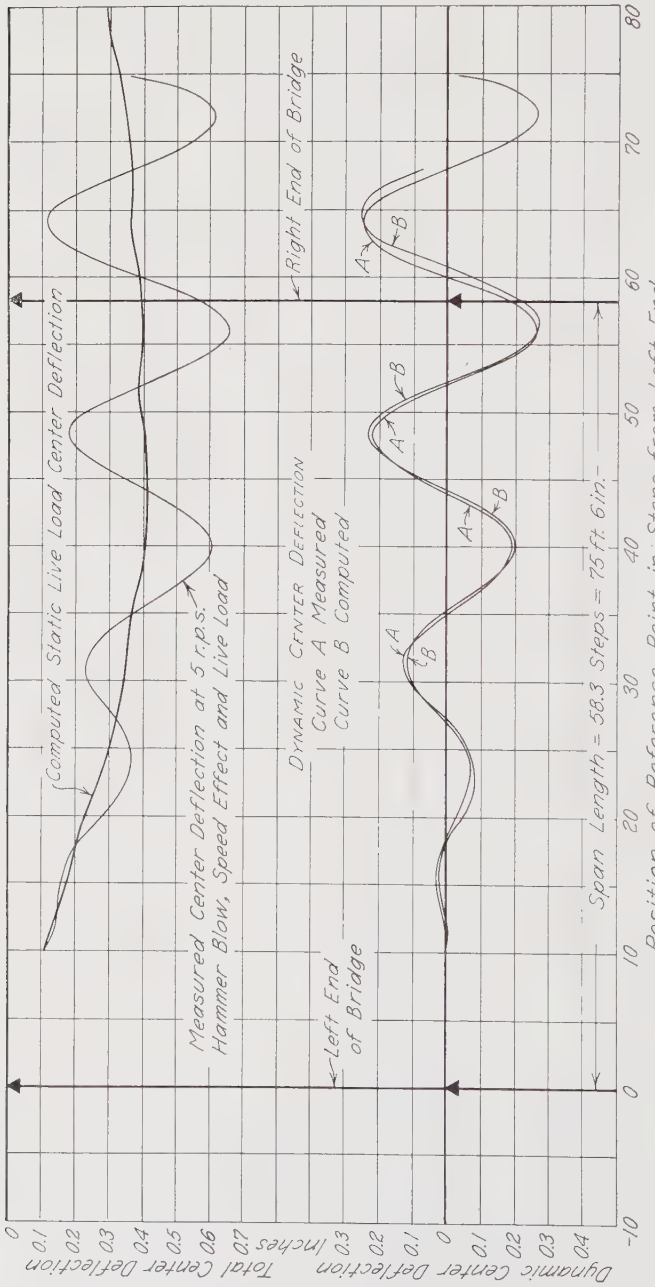


FIG. 27. MEASURED AND COMPUTED VALUES OF CENTER DEFLECTION OF BRIDGE 21 DUE TO HAMMER BLOW AND SPEED EFFECT; FIVE-AXLE EQUIVALENT TO LOCOMOTIVE J1-D; MEASURED VALUES FROM MODEL TESTS; 5 R.P.S.; NO DAMPING

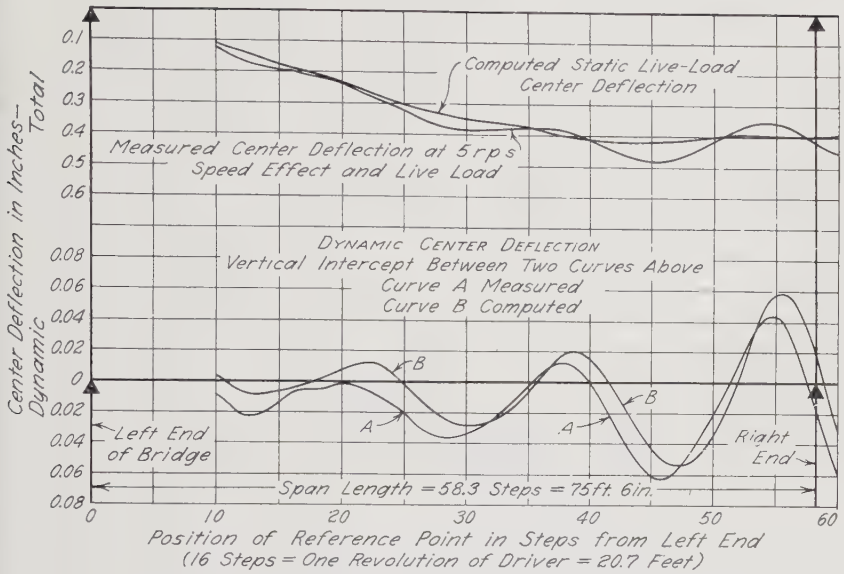


FIG. 28. MEASURED AND COMPUTED VALUES OF CENTER DEFLECTION OF BRIDGE 21 DUE TO SPEED EFFECT; FIVE-AXLE EQUIVALENT TO LOCOMOTIVE J1-D WITHOUT HAMMER BLOW; MEASURED VALUES FROM MODEL TESTS; 5 R.P.S.; NO DAMPING

to be acceptable. This question was studied in two ways: first, by comparing the results by analysis for an equivalent single-axle load and for a series of axle loads; and second, by comparing the results obtained experimentally, using an equivalent single-axle load and a series of axle loads.

The term "single-axle equivalent locomotive" may be assumed to imply that the single-axle load plus the hammer blow will cause the same maximum center oscillation as the locomotive. This term may also be assumed to indicate that the single-axle load plus the hammer blow will cause the same maximum center deflection as the locomotive.

If we consider a single locomotive without any cars following, it is possible for a single-axle load to be approximately equivalent for long spans where the length of the locomotive is short as compared with the span length. This may be demonstrated by plotting, for different span lengths, a term in the preliminary data of the step-by-step analysis which is a function of the locomotive data.

The term $A = \Sigma \frac{P}{K} \frac{1}{g} \sin^2 \frac{\pi x_P}{L}$ (see page 34) will serve for this purpose. For a single concentrated load this term becomes

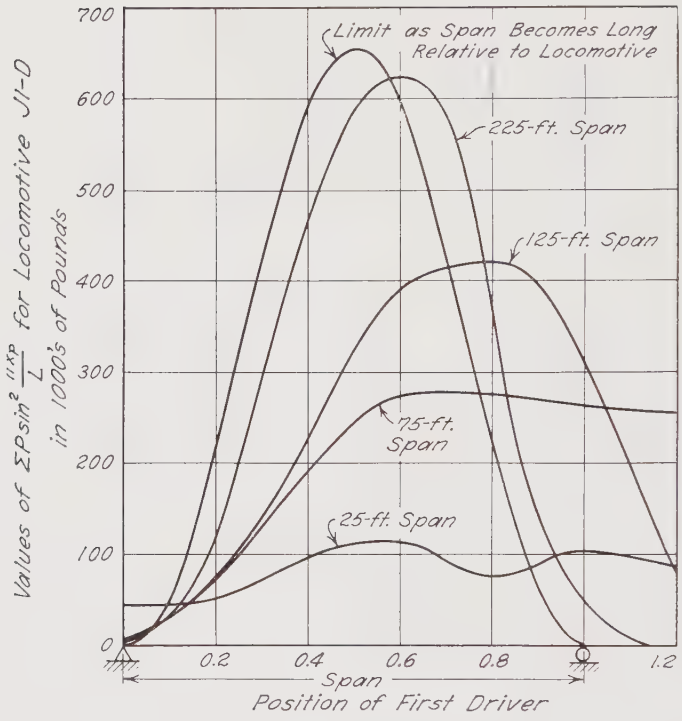


FIG. 29. RELATION BETWEEN DISTANCE OF FIRST DRIVER FROM LEFT END AND VALUES OF $\Sigma P \sin^2 \frac{\pi x_P}{L}$ FOR LOCOMOTIVE J1-D ON BRIDGES OF DIFFERENT SPAN LENGTHS

$\frac{P}{K}\frac{1}{g}\sin^2\frac{\pi x_P}{L}$, corresponding to a sine squared curve. When the value of $A = \Sigma\frac{P}{K}\frac{1}{g}\sin^2\frac{\pi x_P}{L}$ approximates a sine squared curve, then the locomotive can be represented by a single-axle load.

The values of $\Sigma P \sin^2 \frac{\pi x_P}{L}$ are plotted on Fig. 29 for locomotive J1-D and for different span lengths. From this figure it is apparent that a single-axle equivalent can be used in place of a single locomotive for spans over 200 feet long. It does not follow from this analysis, however, that two single-axle equivalents can be used for two locomotives located as required by the British 1928 report—even for a 200-foot bridge. It should be noted that the effect of a train following was not considered.

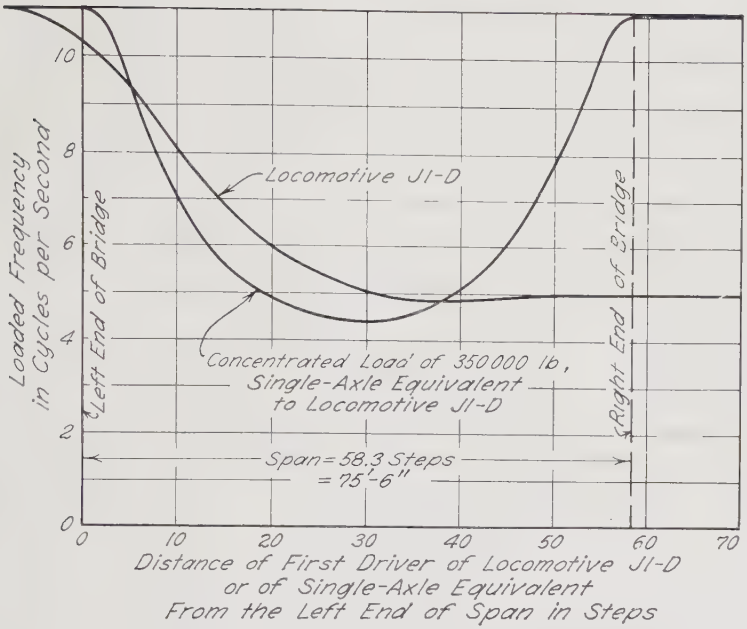


FIG. 30. LOADED FREQUENCY CURVES FOR BRIDGE 21 LOADED WITH LOCOMOTIVE J1-D AND WITH SINGLE-AXLE EQUIVALENT OF J1-D

In the choice of a single-axle equivalent, consideration should be given to the frequency relation of the bridge when loaded with a locomotive and when loaded with a single-axle load. Figure 30, above, shows the relationship between the loaded frequency and position of load for locomotive J1-D, and for a concentrated load of 350 000 pounds. This figure shows the marked difference between the loaded-frequency curves for the two locomotives. This difference will decrease as the span length increases because of the reduction in the ratio of the weight of the locomotive to the weight of the bridge, and also because of the reduction of the ratio of the length of the locomotive to the length of the bridge.

When locomotive J1-D is in the position to cause maximum moment for bridge 21, the center deflection is 0.419 inches and the loaded frequency is 5 cycles per second. The single-axle load at the center which will cause the same deflection as the locomotive is 350 000 pounds, and the loaded frequency in that position is 4.4 cycles per second. The single axle load at the center which will cause the same

loaded frequency as the locomotive is 274 000 pounds, and the deflection for this load is 0.328 inches.

The discussion in the preceding paragraphs indicates that the use of a single-axle equivalent for short spans is open to question.

The computed dynamic deflection of the center of bridge 21 due to combined hammer blow and speed effect is given by the diagrams of Fig. 13. These curves, which represent computed values obtained by the step-by-step analysis, indicate that, for bridge 21, the maximum dynamic center deflection was very nearly the same for the five-axle equivalent as for locomotive J1-D, and that the deflection was considerably greater for the single-axle equivalent than for either the five-axle equivalent or locomotive J1-D.

The experimental deflections of the center of the model bridge due to combined hammer blow and speed effect of the single-axle and five-axle model locomotives is given by the Tables 2, 3, 4, and 5. Tables 2 and 4 give the following results for a speed of 5 r.p.s. with the phase angle set to produce the maximum effect.

	Single Axle	Five Axles
Maximum center deflection	0.610 inches	0.768 inches
Maximum semiamplitude	0.441 inches	0.399 inches

The data presented indicate that equivalence is possible for long bridges with a single locomotive without a train. For short bridges the equivalence of a single load does not seem possible. Equivalence may be possible with five axles where it is not possible for a single axle.

32. *Determination of Center Oscillations of Bridge by Theory of Forced Vibration of Spring With Velocity Damping.*—The conclusion of the Bridge Stress Committee of Great Britain was that the problem of a single-axle load moving across a bridge could be simplified to that of a single-axle load stationary in the middle of the bridge with the wheel slipping so that the hammer-blow force would be applied at the same point. This simplification, the British 1928 report stated, produced very little error in the computed oscillations, and was on the side of safety. The mathematics of this problem is that of the forced vibration of a spring with velocity damping. For several reasons, this conclusion would seem to be in error, particularly for short bridges.

(a) The hammer-blow force is applied a limited number of times by the locomotive while the spring theory is based on an infinite number of applications of the exciting force. For a driver circumference of 20 feet there will be from 3 to 5 applications of the hammer blow on bridges from 60 to 100 feet in length.

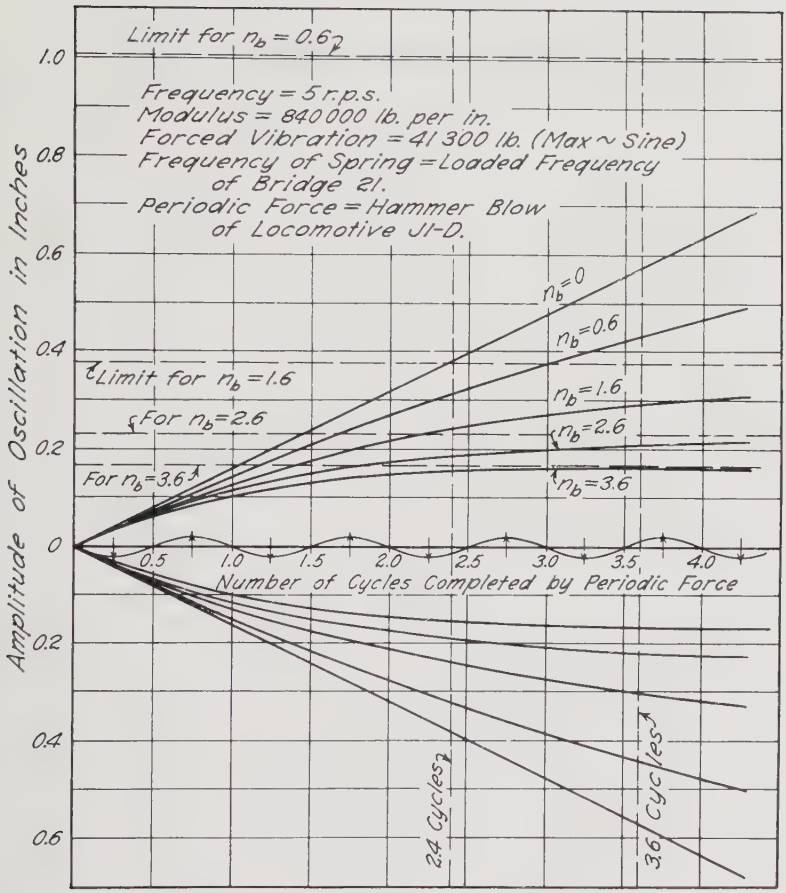


FIG. 31. RELATION BETWEEN NUMBER AND AMPLITUDE OF OSCILLATIONS OF SPRING WITH PERIODIC FORCED VIBRATION AND DAMPING; PERIODIC FORCE SYNCHRONOUS WITH SPRING AND DAMPING PROPORTIONAL TO VELOCITY; INITIAL DEFLECTION AND VELOCITY ZERO

The relation between the number and the amplitude of the oscillations of a spring with forced vibration and damping is shown by the curves of Fig. 31. The initial deflection is zero, the exciting force is synchronous with the spring, and the damping is proportional to the velocity. Curves are given for the various degrees of damping represented by values of n_b of 0.0, 0.6, 1.6, 2.6, and 3.6. The limiting value of the amplitude for an infinite number of cycles is indicated by a separate horizontal line for each degree of damping. These lines are the horizontal asymptotes of the curves for the various degrees of

damping. The modulus of the spring is the modulus of bridge 21 and the exciting force is the hammer blow of locomotive J1-D at 5 r.p.s.

The span and driver circumference of the bridge and locomotive are 75.5 and 20.7 feet, respectively, which limits the application of the hammer blow to 3.6 revolutions. The vertical line at the right of the figure indicates this limit. It is evident that only under very heavy damping would 3.6 revolutions suffice to build up the maximum amplitude. For a smaller damping factor, the amplitude of the oscillations would be less than the value given by the spring theory because there would not be enough applications of the hammer blow to build up the oscillations to the limiting value.

It should be noted that the conditions here are ideal for building up amplitude, since there is synchronism between the frequency of the vibrating system and the exciting force, and full effectiveness of the force in causing deflection. These are the two factors discussed in (b) and (c) following. Any deviation from these ideal conditions would decrease the amplitude of the oscillations produced by a finite number of applications of the force.

(b) The natural frequency of the bridge is changed by the weight of the locomotive and varies continuously during its passage. In the spring theory the frequency remains constant. The variations in the loaded frequency of bridge 21 when loaded with locomotive J1-D and when loaded with the single-axle equivalent are given by the curves of Fig. 30.

The results of the analysis for a simple spring with a forced vibration with damping is shown on Fig. 32. These curves show the semi-amplitude of oscillation as ordinates to the frequency of the exciting force as abscissas. The natural frequency of the system is 5 r.p.s. and the magnitude of the exciting force is equal to the hammer blow of locomotive J1-D. These curves show the marked effect on the oscillation of the relationship between the natural frequency of the system and the frequency of the exciting force. It is only over a narrow range near synchronism, approximately from 4.0 to 6.0 r.p.s., that large oscillations occur. The theory shows that in this narrow range near synchronism, the exciting force is 90 deg. out of phase with the oscillation of the system. Outside of this region the exciting force and the system are in phase. An examination of Fig. 30 shows that with a hammer-blow frequency of 5 r.p.s. there would be approximately $\frac{2}{3}$ of the span length within the 4.0 to 6.0 r.p.s. range, and thus about $\frac{2}{3}$ of 3.6, or 2.4 revolutions within this narrow range of synchronous behavior. It should be noted that the amplitude does not immediately spring to

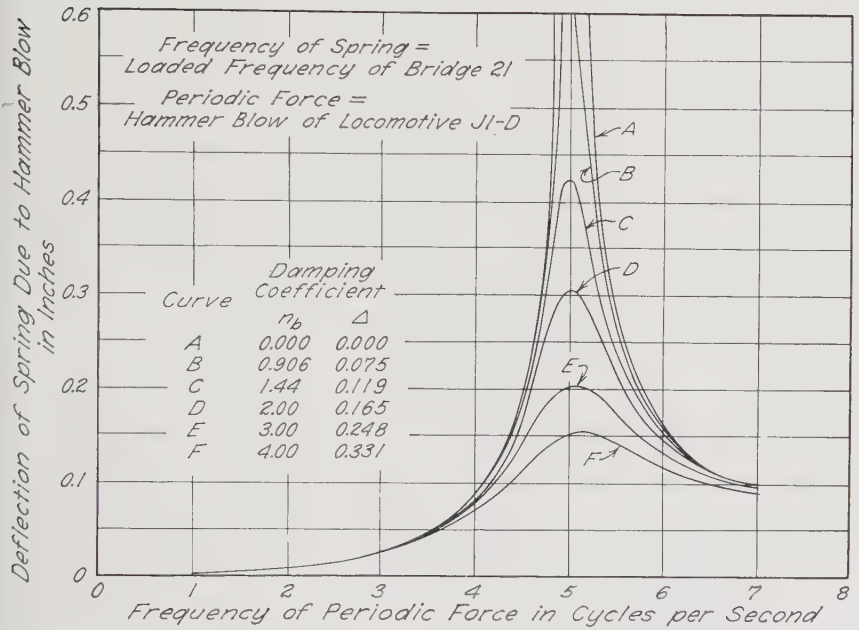


FIG. 32. RELATION BETWEEN MAXIMUM SEMI-AMPLITUDE OF A SPRING AND FREQUENCY OF PERIODIC FORCE; INFINITE NUMBER OF OSCILLATIONS; DAMPING VARIED WITH VELOCITY; FREQUENCY OF SPRING = LOADED FREQUENCY OF BRIDGE 21; PERIODIC FORCE = HAMMER BLOW OF LOCOMOTIVE J1-D

infinity when synchronism is established, but builds up rapidly according to the curves shown on Fig. 31.

(c) When a locomotive passes across a bridge the hammer-blow force is applied, successively, at various points over the full length of the span. The hammer-blow force at the quarter point does not cause as much center deflection as the same hammer-blow force at the center.

An estimate can be placed on this factor by computing the oscillation due to the locomotive hammer-blow force crossing bridge 21, assuming the frequency of the bridge to be constant and equal to 5 r.p.s., and the frequency of the hammer blow to coincide with the constant bridge frequency. Then the oscillation due to the same hammer-blow force applied at the center only, with the same frequency, is computed. An estimate shows the deflection for the moving hammer blow to be 0.37 inches, and for the stationary hammer blow at the center 0.61 inches, the number of applications of the hammer blow being 4 in each instance.

This analysis shows that the distribution of the hammer blows on the span has a considerable effect on the oscillations which they produce.

The three factors discussed in (a), (b) and (c) combine to reduce the amplitude of oscillation in the following manner:

	Deflection, No Damping, in.
Spring theory: (a) Infinite number of revolutions of hammer blow at 5 r.p.s. (b) Constant spring frequency of 5 r.p.s. (c) Full effectiveness of hammer-blow force	Infinite
Spring theory: (a) 3.6 revolutions of hammer blow at 5 r.p.s. (b) Constant spring frequency of 5 r.p.s. (c) Full effectiveness of hammer-blow force	0.57
Bridge with single-axle equivalent locomotive crossing: (a) 3.6 revolutions of hammer blow at 5 r.p.s. (b) Variable bridge frequency (Fig. 30) (c) Reduced effectiveness of hammer blow because of position on bridge	0.44
Bridge with locomotive J1-D crossing: (a) 3.6 revolutions of hammer blow at 5 r.p.s. (b) Variable bridge frequency (Fig. 30) (c) Reduced effectiveness of hammer blow because of position on bridge	0.40

33. *Effects of Smoothly-Rolling Load.*—The effects of a smoothly-rolling load were studied both analytically and experimentally. The results were discussed in Sections 27 and 28. Tests were made using a single-axle model and a five-axle model, and the excellent data obtained provided insight into this dynamic phenomenon. The dynamic effect of a smoothly-rolling load may be subdivided into three parts as follows:

(a) Centrifugal effect of the wheels following the curved path of the deflected bridge; this results in a general increase in the bridge deflection as a whole.

(b) The oscillations due to the sudden application of the wheel loads as they come on the bridge; the wheels impart a series of downward impulses to the bridge which cause it to oscillate.

(c) In a bridge with a floor system of stringers and floor beams a downward impulse is given to the bridge by each wheel as it crosses the floor beam. An oscillation of considerable magnitude can result

from this impulse, if the spacing of floor beams and wheels and the velocity is such as to produce cumulative effects.

These features are discussed in greater detail in the following paragraphs.

(a) Figures 7, 11, 21, 23, and 28 show the center deflection due to speed effect. Centrifugal effect results in a general increase in the static live-load deflection which causes the oscillations evident in the figures mentioned to be displaced below the horizontal zero line. The amount can be found by drawing two envelope curves, one at the top and the other at the bottom of the peaks of the oscillation, and then drawing an average line between them which will represent the centrifugal effect.

The maximum center deflections due to centrifugal effect obtained from these figures in this manner are all small. From Fig. 21 the centrifugal deflection for a single-axle load at 6.25 r.p.s. is 0.01 inch, and from Fig. 23-b the deflection for five axles at 5.7 r.p.s. is 0.024 inch. Expressed as percentages of the maximum static live-load deflection these values would be 2.4 and 5.7, respectively.

The centrifugal effect in the step-by-step analysis is represented by the term $-d\frac{\pi^2}{L^2}(\dot{x}_P)^2 \sin\frac{\pi x_P}{L}$ of Equation (3c), page 29. The effect of this term on the computed speed effect for five axles is shown by curve B of Fig. 28.

(b) The oscillations due to a series of smoothly-rolling loads are very interesting. Experimental curves for single-axle and five-axle models are shown in Figs. 21 and 23, respectively, and both experimental and computed curves in Fig. 28.

The test of the single-axle model is of qualitative significance only, since a single axle with a weight half that of the locomotive cannot be representative of the prototype. It does, however, show that a load rolling suddenly onto a bridge causes an oscillation of appreciable magnitude.

The single-axle load caused a maximum semiamplitude of 0.035 inch due to the sudden application of a heavy-rolling load, as shown by Fig. 21. There can be no cumulative effect since there is only one axle load. The left end of the deflection diagram indicates what is happening. This portion of the diagram shows the center deflection at the instant when the load comes onto the bridge. The load is traveling at the high speed of 130 feet per second. The load, coming onto the bridge quickly, acquires a large downward acceleration and the inertia of the bridge resists this acceleration. As a result, the deflection of

the center when the axle is near the left end is less than the static live-load deflection for the corresponding positions of the axle. The forces acting on the wheel and bridge are not balanced, and as a result the two bodies acquired kinetic energy, hence the oscillations.

The five-axle model load consisted of much smaller individual loads than the single-axle load, and the series of wheels made cumulative oscillations possible. The magnitude of these oscillations, and the manner in which they increase in amplitude clearly indicate accumulation due to the series of impulses.

The proportions of the model were such that there was a series of wheels at approximately equal distances, and the spacing was approximately equal to the circumference of the driver. Consequently, the model produced a series of downward impulses on the bridge which were equally spaced with respect to time. At velocities of 5 and 6 r.p.s., the frequency of the impulses was approximately the same as the loaded frequency of the bridge, so that cumulative oscillations were possible.

The curves of Fig. 23 show an increase in amplitude up to a maximum of 0.06 inch at 5.7 r.p.s., and then a decrease at 6.33 r.p.s. This result is reasonable when one considers the loaded frequency as shown by the curve on Fig. 30.

It is interesting to note that by varying the wheel spacing one could produce a series of impulses that would coincide with the variable loaded frequency of the bridge.

The curves of Fig. 23 show that the oscillations are affected by damping because of their high frequency and consequent velocity.

The increase in center deflection due to a smoothly-rolling load, which includes centrifugal effect plus the oscillation, is shown by the lower curve of Fig. 24. The maximum deflection due to the moving load minus the maximum static live-load deflection is plotted against velocity in r.p.s. This deflection is approximately two-thirds oscillation and one-third centrifugal. The maximum value is 0.07 inch, or 17 per cent of the maximum static live-load deflection.

It is evident from the analysis of the prototype shown on Fig. 7, that the thirteen axles would not cause as large oscillations as the five axles of the model shown on Fig. 11. It cannot be overlooked, however, that it is possible to group the wheels of the prototype in such a way that they will produce oscillations. A group of wheels closely spaced might act together to cause a joint impulse. In this way the thirteen-wheel locomotive might have an effect similar to that of

the five-wheel model. The group effect of the three axles of each of the two trucks of the tender is evident in Fig. 7.

These tests indicate that centrifugal effect is of no great importance for the set of conditions tested, and probably this is true in general. The oscillations obtained are interesting, and will prove, undoubtedly, the most significant part of speed effect.

(c) No experiments were made with a flexible floor system. A 100-foot bridge with five 20-foot panels would receive four impulses with a frequency of 4 cycles per second from a single load with a velocity of 80 feet per second. Since 4 cycles per second is the approximate loaded frequency of such a bridge, cumulative oscillations are possible. The discussion in part (b) indicates that tests for this effect would produce some very interesting results, and such tests could easily be made.

34. *Effect of Phase Angle of Hammer Blow Upon Oscillations.*— The effect of the phase angle of the hammer blow on the oscillations was investigated in the A.R.E.A. 1911 tests. No definite conclusions could be drawn from the results of the field tests. No other significant information is available.

The effect of the phase angle of the counterweight upon the maximum center deflection for the single-axle model and for the five-axle model, is shown by the curves of Figs. 20 and 22. Data on both the increase in center deflection and the amplitude of oscillation are given on Tables 2 and 4.

It is apparent that the phase angle is significant for short spans. There are two factors involved: one, the relation of the position of the maximum downward value of the hammer blow to the center of the bridge; and the other, the number of hammer blows within that part of the span length where there is approximate agreement between the loaded frequency of the bridge and the frequency of the hammer blow.

The data for the five-axle model are the most significant. The curves of Fig. 22 show a considerable variation in the increase of center deflection due to the change of the phase angle. These deflections, which are for hammer blow plus speed effect, show the largest difference in the increase in center deflection to be 0.111 inch, or 26.5 per cent of the maximum static live-load deflection. Additional data presented show that the increases in center deflection due to hammer blow alone can differ by 0.039 inch or 9.3 per cent of the maximum static live-load deflection.

In the tests with five axles the phase angle of the counterweight to give maximum center deflection was the same as that for maximum

semiamplitude at velocities of 4.15, 5.00 and 5.72 r.p.s. At the highest velocity of 6.33 r.p.s., however, the phase angle was different. The position of the counterweight to cause maximum center deflection was used in the remainder of tests b and c.

In the tests without damping, the position of the reference point when the maximum center deflection occurred was well past the position for maximum moment (0.75L for the five-axle model) at the velocities of 5.00 and 5.72 r.p.s. In the tests with damping the maximum center deflection occurred with the model approximately in the position for maximum moment. The one exception, velocity 5 r.p.s. and $n_b = 0.40$, gave a maximum center deflection for hammer blow plus speed effect and live load with the reference point at 1.08L, and a maximum center deflection due to hammer blow plus live load with the model at 0.81L.

It can be said for the five-axle model that the maximum center deflection and maximum semiamplitude occurred with the same phase angle of the counterweight. Both of these deflections occurred with the model approximately in the position for maximum moment and for maximum static live-load deflection.

35. *Relationship Between Results Obtained by Step-by-Step Analysis and by Tests of Models.*—Tests of models were made to demonstrate the possibilities of model testing as a tool for determining the impact on railway bridges. Parallel investigations were made with the step-by-step analysis and with tests of models of the single-axle and the five-axle equivalents of locomotive J1-D. Agreement between the results obtained by the two methods would be evidence that both methods gave satisfactory results.

The dynamic deflection at the center of bridge 21, caused by the single-axle equivalent of J1-D as it rolled across at 5 r.p.s. is given by the two curves, A and B, at the bottom of Fig. 19. Curve A represents results obtained from tests of the model and Curve B represents values obtained by the step-by-step analysis. These curves show that the values of the dynamic center deflection obtained by tests and by the step-by-step analysis agree closely. The greatest difference occurs as the model is about to go off the bridge, at this position the static live-load deflection was very small.

The dynamic deflection at the center of bridge 21 caused by the five-axle equivalent of locomotive J1-D as it rolled across at 5 r.p.s. is given by the diagrams of Fig. 27. Curve A represents the results of the model tests and Curve B represents the results of the step-by-step analysis.

The same five-axle equivalent locomotive without counterweight was tested and the results compared with those of the analysis. The results are shown on Fig. 28. Curve A represents the results of the model tests, and Curve B represents the results of the step-by-step analysis.

The degree of agreement obtained between the results obtained with the step-by-step analysis and those obtained by the tests of models indicate that both methods give results acceptable in determining the dynamic deflection due to combined hammer blow and speed effect, and that due to speed effect alone.

36. *Comparison of Results Obtained in This Investigation With Results Obtained by Other Investigations.*—In the A.R.E.A. 1911 report, the significance of the hammer blow and the cumulative oscillations of the bridge were first recognized. They were studied experimentally by means of field tests, and analytically to obtain a relationship between the frequency of the hammer blow and the loaded frequency of the bridge.

The British 1928 report made use of a fully-developed theory for the dynamic center deflection. This theory was checked by comparison with field tests, and then used to formulate an impact allowance.

The A.R.E.A. 1936 report made use of the theory of the British report to derive the damping coefficients for American bridges. The same theory was used to obtain an impact allowance. The utilization of the analysis in these reports is discussed in Chapters III and IV.

The data obtained by the tools developed in this bulletin are similar to those used by the A.R.E.A. 1936 and the British 1928 reports. These data are presented in the form of computed and experimental center-deflection curves, and curves relating dynamic deflection and frequency of hammer blow. In addition, new material on speed effect was obtained.

A comparison will be made between the data in this bulletin and the data in the reports. Some of the difficulties involved in the use of these data to derive rules for impact allowances will be discussed.

The verification of theory with field tests in the British 1928 report was made with long span bridges and short, light locomotives. A question still remains as to whether or not the measured and computed values of the center deflection would have been in fair agreement if short spans and heavy locomotives had been used. This question was discussed in Section 31. It is certain that the dynamic deflections for a single-axle load, the loading used in the British theory, do not

conform with those for a series of axles, both equivalent to the same locomotive. The step-by-step analysis makes it possible, however, to compute accurately the deflection due to a series of axles. Data on bridge damping and on the spring action of the locomotive are still incomplete. These data must be obtained before computations of bridge deflections can be made which would be acceptable as representing the true deflections.

In the British 1928 report, the analysis for a single axle crossing the bridge was simplified to that of a stationary single axle on the bridge with the wheel skidding. The solution of this problem is identical with that of the forced vibration of a spring with damping. This same theory was used in the A.R.E.A. 1936 report. The usual method of presenting the results of this analysis is to draw a diagram showing the maximum dynamic deflections for various amounts of damping as ordinates, to the r.p.s. of the hammer blow as abscissa. It will be interesting to compare these theoretical curves for a spring with similar curves obtained experimentally for a model bridge and locomotive. A set of curves based on the theory of a damped spring is shown on Fig. 32, page 111. These curves should be compared with those of Figs. 25 and 26 which were obtained experimentally with the five-axle model. The hammer blow and modulus of the spring were the same as those of locomotive J1-D and bridge 21. There is evidently very little similarity between the spring and the bridge curves.

An explanation of the difference between the ordinates of the curves of Figs. 25 and 26 and those of Fig. 32 for the same amount of damping is given in Section 32.

The curves of Figs. 25 and 26 show the following interesting points: The phenomenon of cumulative oscillations is evident; the curves rise sharply from about 4 r.p.s. and then show a decrease as the velocity increases. These curves do not show a narrow peak as do the spring curves, but a range of velocities from 5 to 6 r.p.s. over which maximum values occur. The curves do not show a sharp dropping off as the velocity increases past the maximum deflection. This is due to the variable loaded frequency of the bridge and locomotive, and also to the fact that the hammer blow increases as the velocity increases.

A comparison of Figs. 25 and 26 shows that there is a change in the deflection when the speed effect is subtracted. This is true even at the lower velocities. When a comparison is being made between field test data and computed data, curves similar to those of Fig. 26 should be

used since these deflections are due to hammer blow plus speed effect as are the field deflection data.

The theory presented in the British 1928 report was used in the A.R.E.A. 1936 report to obtain curves of maximum dynamic deflection as ordinates to r.p.s. as abscissa. These curves are identical with those of Fig. 32. The results of field tests were plotted on the same diagram and the curve which best fitted the field data was selected. In this manner the damping coefficient of the bridge was taken from that of the curve selected. It is evident that the damping coefficients obtained in this manner will represent the effect of all the factors that limit the dynamic effect, and not merely the loss of energy due to friction, that is, bridge damping. It is possible that other factors may be involved, such as a spacing of the locomotive axles which would influence the speed effect, or a circumference of the drivers which would determine the number of applications of the hammer blow. If these factors and others listed on page 122 have appreciable influence, the damping coefficients that would be obtained for a particular bridge would depend upon the characteristics of the locomotive used in the test, and different values would be obtained from different locomotives.

The effect of damping upon the various parts of the live-load deflection is interesting. The diagrams of Fig. 23 show that damping has little, if any, influence on centrifugal effect, since the deflection is about the same for $n_b = 0$ and $n_b = 2$. This is reasonable since the damping is proportional to the vertical velocity of the bridge, and the centrifugal effect is a relatively slow motion as compared with the oscillations due to hammer blow or speed effect. The effect of damping on the oscillations due to speed effect is marked. These oscillations do not die cut, however, as has been suggested by the theory of the Bridge Stress Committee, since they are caused by a series of impulses which constitute a forced vibration that continues while the wheels are coming onto the bridge.

The curves of Figs. 25 and 26 for hammer blow almost coincide up to 4 r.p.s., and this indicates the slight effect of damping on the deflection up to this velocity. In order to obtain the bridge damping from these curves by the method employed in the A.R.E.A. 1936 report it would be necessary to have field deflection data for velocities between 5 and 6 r.p.s. where the curves are separated. A 20-per-cent error in the field deflection data at 4 r.p.s. could make an error of 100 per cent in the maximum deflection indicated by the selected curve.

Figure 33 shows the maximum dynamic deflections for each of four

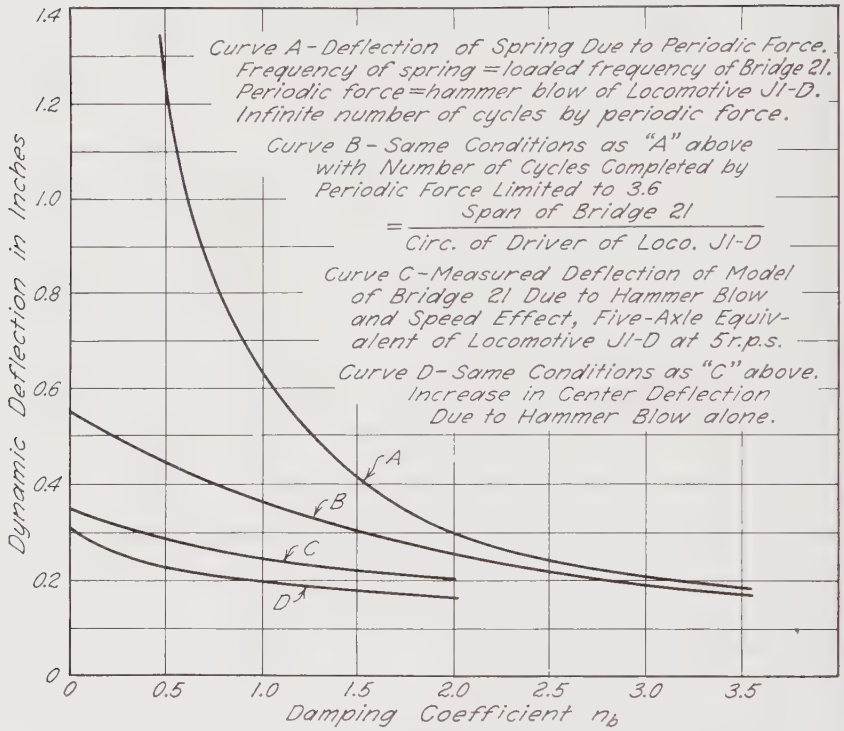


FIG. 33. COMPARISON OF DYNAMIC DEFLECTIONS OF DIFFERENT SYSTEMS DUE TO SAME PERIODIC FORCE

systems A, B, C, and D plotted against the damping coefficient n_b. These curves show the considerable difference between the maximum deflections for the spring and for the bridge. Curves A and B converge rapidly, as would be expected from Fig. 31, which shows that for heavy damping only a few revolutions of the forced vibration are necessary to build up the maximum amplitude. Curves C and D appear to maintain an approximately constant percentage difference.

Where the damping is heavy, large changes in the damping make little change in the deflection. Therefore, accurate determinations of n_b by the use of locomotive field-deflection measurements would be difficult.

The apparent magnitude of the damping force as indicated by theory and tests is significant. In order to obtain a measure of the damping force, the distributed force will be reduced to an equivalent centrally-applied force which will cause the same center deflection

(see page 47). This force would be $\frac{192wL}{\pi^3 g} n_b v = 26\,200 n_b v$ pounds. The results of model tests given on Fig. 26 show a maximum dynamic deflection of 0.18 inch at 5.8 r.p.s. for a value of $n_b = 2$. Assuming simple harmonic motion, an amplitude of 0.18 inches would be associated with a velocity of 6.56 inches per second. These values of n_b and v give for the equivalent central force

$$26\,200 \times 2 \times \frac{6.56}{12} = 28\,700 \text{ pounds.}$$

The maximum damping force for the prototype would be 28 700 pounds as compared with a hammer blow of 55 000 pounds and the total weight of locomotive J1-D, 650 000 pounds.

This is a very large damping force, and raises the question as to the manner in which the energy is dissipated. A force of this magnitude would do about $(\frac{2}{3} \times 0.18 \times 28\,700)\ 2 = 6890$ inch pounds of work per cycle with a semiamplitude of 0.18 inches. With a frequency of 5.8 per second the horsepower would be 6.1.

A figure can be placed on the loss of energy due to hysteresis. The width of the hysteresis loop is given in Morley's "Strength of Materials"* as about $\frac{1}{300}$ of the total strain. The stress at the center of the span for ± 0.18 inch deflection would be equivalent to ± 3100 pounds per square inch, or a total variation of 6200 pounds. Work per cycle is equal to the area of the hysteresis loop, or, $\frac{2}{3}\left(\frac{6200}{E} \times \frac{1}{300} \times 6200\right) = 0.00285$ inch pounds per cycle, per cubic inch. An estimate of 0.7 of the total steel in the bridge at this stress is 80 000 pounds. This gives the total work per cycle for the bridge as 805 inch pounds.

An estimate of the work done by the sliding of the end bearings can be made for comparison with the work done by the hammer blow. The maximum end reaction for bridge 21 would be about half the weight of the locomotive and bridge, or 400 000 pounds. The distance of sliding motion due to a center stress of $\pm$ 3100 pounds per square inch is about

$$2 \times \frac{2}{3}\left(\frac{3100 \times 75.5 \times 12}{E}\right) = 0.125 \text{ inches.}$$

*Morley's "Strength of Materials," 8th Edition, page 99.

Assuming a coefficient of kinetic friction of 0.1 for steel on steel the total work per cycle is

$$2\,(0.1 \times 400\,000 \times 0.125) = 10\,000 \text{ inch pounds.}$$

These figures are offered only as an indication of the order of magnitude.

Inglis, who developed the theory for the British 1928 report, ascribes a considerable amount of damping to continuity of the track at the abutment. This resistance is in the form of couples resisting the angular deflection of the bridge ends, and he states that this resistance is very important for short span bridges.

The damping coefficients recommended by the British 1928 report are much smaller than those recommended by the A.R.E.A. 1936 report. A comparison for bridge 21 gives $n_b = 0.6$ or $\Delta = 0.05$ from the British report; and $n_b = 3.9$ or $\Delta = 0.32$ from the A.R.E.A. report. It must be remembered that there are great differences between the bridges and locomotives of the two countries. The British report states that the damping coefficients were obtained by field experiments from residual oscillations. The exact procedure is not clearly stated, but from the information given, it appears that the coefficients were derived from the rate at which the vibrations set up by the passage of the locomotive died out. The obvious need is for field tests to determine damping coefficients. These should be made with the bridge loaded, and by the use of an oscillator or some other apparatus which determines damping directly.

The field tests of the A.R.E.A. 1936 report obtained a maximum semiamplitude of deflection of 0.15 inch for bridge 21 and locomotive J1-D. The author believes that bridge damping due to friction is not sufficient to account for this limitation of deflection. A number of factors can be listed which might be termed damping if the word is generalized to mean anything which will limit the oscillation of a railway bridge and prevent the infinite oscillation associated with the forced vibration of a spring. Some of these factors are listed below:

(1) Friction of the bearings of the bridge
(2) Friction within the material of the bridge systems
(3) Interference by the vibrations of individual bridge members
(4) Friction of ballast
(5) Interference due to rough or uneven track
(6) Slip of the locomotive drivers

(7) Variation of frequency of the bridge due to weight of the locomotive

(8) Limitation by the span length of the number of applications of the hammer blow

(9) Distribution of the hammer-blow force over the span length

(10) Spring action of the locomotive

(11) Friction of the locomotive springs

(12) The fact that the locomotive is not a series of independent loads, but a large unit composed of the cab, boiler, firebox, and engine frame supported on the locomotive axles and wheels through an equalizer system.

Item (10) seems to hold the most promise for investigation. The analysis for bridge 21 and locomotive J1-D on pages 52 to 60 indicates that some motion is possible and even probable. The vertical acceleration of the center of the bridge is given in column (3) in inches per second per second.

The maximum accelerations and deflections (less $\Delta_{D.L.} = 0.145$) are as follows; the position of the locomotive is given in terms of the distance of the first driver from the left end.

Position (L=Span)	Acceleration of Center		Total Center Deflection in.	Dynamic Center Deflection in.
	in. per sec. per sec.	times g		
$0.27L$	33.0	0.08	0.154	−0.030
$0.41L$	70.4	0.18	0.347	0.064
$0.55L$	115.2	0.30	0.245	−0.108
$0.70L$	177.5	0.46	0.590	0.185
$0.84L$	233.4	0.60	0.178	−0.224
$0.96L$	279.7	0.72	0.683	0.291
$1.10L$	288.9	0.75	0.095	−0.286
$1.25L$	301.2	0.78	0.646	0.302

Taking, for example, the position $0.41L$, these figures indicate that if the force necessary to overcome resistance to the motion of the axles relative to the sprung weight of the locomotive were less than 0.18 of the sprung weight, then the axles would move relative to the sprung weight. Since this force has been estimated as approximately 0.15, this motion would have occurred under the test conditions of $n_b = 0$ and 5 r.p.s. Such a motion would throw the action of building up the amplitude out of phase and limit the amplitude of oscillation.

The amplitude of the center deflection at the particular point selected above was only 0.065 inches or 16 per cent of the static live-load center deflection.

X. Conclusions

37. *Conclusions.*—Only one bridge and locomotive have been investigated in this bulletin, and therefore, no sweeping statements will be made on impact in general. As has been stated before, the author's principal objective was to develop reliable and practical tools of an experimental and analytical nature to be used in planning field tests and to aid in the interpretation of the results. It is believed, however, that the following statements have been justified by the analyses and discussions contained in the previous chapters.

(1) In the analysis of bridge oscillations due to hammer blow, each locomotive can be represented by an equivalent single-axle load for spans over 200 feet in length. For spans below 200 feet it is necessary either to use a five-axle equivalent locomotive, or to introduce into the analysis the axle loads and spacings of the locomotive. The step-by-step method is an accurate and practical method of analyzing the motion of a bridge due to a series of axle loads.

(2) The factors listed below have an important bearing on the oscillations due to hammer blow on short and medium length bridges.

(a) Variation of the loaded frequency

(b) Limitation by the span length of the number of applications of the hammer blow

(c) The effect of the distribution of the hammer-blow force over the span length.

These factors cause large differences between the results obtained by the use of the spring theory and those obtained by the equivalent single-axle load.

(3) Bridge damping factors obtained indirectly by the use of the spring theory will be affected by bridge and locomotive characteristics as well as by true damping, which is usually considered to be a loss of energy due to internal or external friction. It seems probable that bridge damping for short spans is much smaller than has been indicated by the A.R.E.A. 1936 report. The limitations of amplitude involves spring action of the locomotive, as proven by the accelerations indicated in the analysis made by the author.

(4) Centrifugal effect is relatively small, being of the order of 5 per cent of the total dynamic effect.

(5) Oscillations due to the sudden application of the wheel loads

do not die out because of damping, as previous theory suggests. Synchronous impulses may result from the relation between the axle spacing, the locomotive speed, and the loaded frequency of the bridge, which will result in a considerable amplitude of oscillation. This is significant for locomotives either with or without hammer blow.

(6) The phase angle of the hammer blow is of importance in determining the maximum dynamic effect due to hammer blow.

(7) On short and medium length spans, large oscillations may occur over as much as a 15 m.p.h. range of velocity.

(8) The curves relating amplitude of oscillation due to hammer blow to the r.p.s. of the locomotive driver are very different from comparable curves for the forced vibration of a spring with damping.

(9) Analytical and experimental tools developed in this investigation make it feasible to investigate quickly and easily the importance and influence of the many factors that enter into the impact in railway bridges. This method makes it possible to plan field tests efficiently and should result in reducing the number of field tests, with a resulting economic advantage. For example, an examination of Fig. 24, page 93, shows that field tests below a speed of 4.5 r.p.s. would be useless, and that a speed of 5.5 r.p.s. should be used. Further, it is seen that the phase angle of the hammer blow should be recorded, and if it is not in the position for maximum dynamic effect, the results could be modified by the use of a curve similar to the experimental curve, Fig. 22, page 86, which shows the relation between phase angle and dynamic deflection. It would be possible, for instance, to use tests of models to determine the dynamic effect of rail joints: a problem which is still in an unsatisfactory stage of solution. A qualitative relationship between deflection and velocity could be obtained by introducing a joint in the rail of the experimental bridge and making a series of tests with a single-axle load. A few field tests, guided by the laboratory tests of models, would provide the necessary scale and an adequate expression for the impact.

The results obtained from the step-by-step analysis and from the laboratory experimental apparatus described in this bulletin sufficiently demonstrate the usefulness of these methods as tools in furthering the study of impact. The tests and analyses indicate that field tests require supplementary investigations, and these methods provide a reliable and economical procedure.

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VITA

Charles Thomas George Looney was born on July 27, 1906 in Liverpool, England. His family came to the United States in 1920 and established their home in Lawton, Oklahoma, where Mr. Looney's secondary education was obtained. He received the degree of Bachelor of Science with honors in Building Construction from the Carnegie Institute of Technology in 1932; and the degree of Master of Science in Civil Engineering from the University of Illinois in 1934. After further graduate work and employment as an engineer, Mr. Looney went to the University of North Carolina at Chapel Hill in September, 1936 as Assistant Professor of Civil Engineering. Since September, 1937 he has been Assistant Professor of Civil Engineering at the State University of Iowa at Iowa City.

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